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J. Electrical Systems 16-4 (2020): 491-497

Regular paper

The Two- Point Capacitance for an Infinite Face Centered Cubic Network

JES
Journal of Electrical Systems

The equivalent (net) capacitance between any two lattice points in an infinite face centered cubic (FCC) composed of capacitors of the same capacitance (C) was calculated using Green's Function (GF) method. Selected numerical values along [100] and [111] directions have been presented. The results obtained showed that there is a symmetric behavior for the effective capacitance calculated along [100], and [111] directions under the transformation of $s_1, s_2, s_3 \rightarrow -s_1, -s_2, -s_3$.

Keywords: Green's Function, Infinite Lattice, Face Centred Cubic Lattice, Identical Capacitors, Net Capacitance

Article history: Received 19 May 2020, Accepted 22 September2020

1. Introduction

Analysing electrical circuits is of importance for both physicists and electrical engineers. The great German scientists Kirchhoff [1] formulated two important laws that can be used to analyse and study electrical networks, the Node Current Law and the Loop Voltage Law. Since Kirchhoff's date this problem attracts many researchers to study and search. In literature one can find many important works that study this problem, for example see [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16]. Basic and main efforts have been paid on electric networks consisting of resistors, where many methods have been applied in studying and analysing infinite and finite electric networks such as: current distribution method [2], [3], [4], random walk method [5], [6], Green's Function (GF) method [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17].

The GF has been used widely in studying many infinite lattices consisting of resistors (e.g., Square, Simple Cubic- SC-, Body Centred Cubic- BCC-, Face Centred Cubic-FCC-, and many others). In these studies, numerical and analytical investigations have been carried out. On the other hand, it is of importance to point our figure to an important method proposed and developed recently. This method is known as recursion-transform approach (or the RT method). The RT method has been used to solve many problems of the resistor network, for example see [18], [19], [20], [21], [22].

Although analysis of capacitor networks is very similar to resistor networks but it is of importance in electric circuit theory. In literature less efforts have been devoted on studying capacitors networks, networks constructed of same capacitors has been studied using the charge distribution method and the lattice GF method [23], [24], [25], [26], [27]. In this work we apply the lattice GF to study a three-dimensional infinite network consisting of identical capacitors (e.g, Face Centred Cubic network-FCC-).

Iwata [28] investigated the GF of the infinite FCC lattice, where he expressed the lattice GF of the infinite FCC lattice at the origin (i.e. $f_o(3;0,0,0)$) in a compressed format in terms of the complete elliptic integrals of the first kind. The lattice GF for cubic lattices at

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any lattice site was also studied by Mano [29]. He expressed it in terms of linear sets of complete elliptic integrals of the first and second kind. In Ref. [30] the lattice GF for infinite SC, BCC, and FCC lattices were expressed rationally in terms of the known values of the lattice GF at the origin for each lattice and π.

The rest of this paper is arranged as follows:

Fundamental relations and preliminaries have been presented in section 2. In Sec. 3, the results quoted in section 2 were applied to the infinite FCC network consisting from identical capacitors. In sec. 4 results are presented with discussion, and finally, we close the paper by a conclusion in Sec. 5.

2. Basic Relation and Preliminaries

In this section, we introduced some important basic relations concerning any perfect infinite d- dimensional lattice composed of capacitors of equal capacitance (C). First of all, all lattice points in the infinite d- dimensional lattice have been assumed to be identified by the position vector $\vec{r}$ defined as [7]:

$$\vec{r} = s_1\vec{b}_1 + s_2\vec{b}_2 + ... + s_d\vec{b}_d \, . \tag{1}$$

Where $s_1, s_2, ..., s_d$ are integers (positive, negative or zero),

and $\vec{b}_1, \vec{b}_2, ..., \vec{b}_d$ are independent primitive translation vectors.

Secondly, the d- dimensional infinite lattice is assumed to be hypercubic (i.e., $\left|\vec{b}_1\right| = \left|\vec{b}_2\right| = ... = \left|\vec{b}_d\right| = b$).

To determine the equivalent capacitance between the lattice sites $(0,0,0)$ and $\vec{r}$ it has been assumed that the potential at the site $\vec{r}'$ be defined as $V(\vec{r}')$, and the charge at the origin is (q), while the charge leaving the site $\vec{r}_o$ is $(-q)$ and zero otherwise. In other words, we can write [23, 24]:

$$q(\vec{r}') = q\left[\delta_{\vec{r}',0} - \delta_{\vec{r}',\vec{r}}\right]. \tag{2}$$

Upon applying Ohm’s and Kirchhoff’s laws one can write:

$$\frac{q(\vec{r})}{C} = \sum_m \left[V(\vec{r}) - V(\vec{r} + \vec{m})\right]. \tag{3}$$

Where $\vec{m}$ are the vectors from the site $\vec{r}$ to its nearest neighbors (i.e., $\vec{m} = \pm\vec{b}_i, \;\; i = 1, 2, ..., d$).

The equivalent capacitance between the sites $(0,0,0)$ and $\vec{r}_o$ reads:

$$C_o(\vec{r}_o) = \frac{-q(\vec{r})}{V(0) - V(\vec{r}_o)}. \tag{4}$$

The potential can be written as:

$$V(\vec{r}) = \frac{1}{C}\sum_{\vec{r}'} G_o(\vec{r} - \vec{r}')q(\vec{r}'). \tag{5}$$

So, upon using Eq. (4) and Eq. (5) the equivalent capacitance between the sites $(0,0,0)$ and $\vec{r}_o$ reads:

$$C_o(\vec{r}_o) = \frac{C}{2\left[G_o(0) - G_o(\vec{r}_o)\right]}. \tag{6}$$

Where $G_o(\vec{r}_o)$ is the lattice GF of the d- dimensional infinite lattice at site $\vec{r}_o$, and $G_o(\vec{0})$ is the lattice GF of the d-dimensional infinite lattice at $(0,0,0)$, for full details we advise those interested in analyzing electrical circuit to refer to [24], [25].

3. Application: two-point capacitance in infinite FCC lattice

Now, we will apply Eq. (6) above to find the two- point equivalent capacitance in an infinite FCC lattice composed of capacitors having the same capacitance C. The position vector for an infinite FCC lattice, which is three-dimensional lattice (i.e., $d = 3$) reads:

$$\vec{r} = s_1\vec{b}_1 + s_2\vec{b}_2 + s_3\vec{b}_3. \tag{7}$$

Also, the lattice GF for the infinite FCC lattice at both $(0,0,0)$, and the site (s_1, s_2, s_3) read respectively as: $f_o(3;0,0,0) = F(3;0,0,0)$ and $F(3;s_1,s_2,s_3)$, as a result Eq. (6), for infinite FCC lattice becomes:

$$C_o(3;s_1,s_2,s_3) = \frac{C}{[f_o(3;0,0,0) - F(3;s_1,s_2,s_3)]}. \tag{8}$$

The lattice GF for an FCC lattice is defined as:

$$F(E;s_1,s_2,s_3) = \frac{1}{\pi^3}\int_0^{\pi}\int_0^{\pi}\int_0^{\pi}\frac{Cos(s_1q_1)Cos(s_2q_2)Cos(s_3q_3)}{E - Cosq_1Cosq_2 - Cosq_2Cosq_3 - Cosq_1Cosq_3}dq_1dq_2dq_3. \tag{9}$$

For $E \geq 3$, and $s_1 + s_2 + s_3$ even ($F = 0$ for $s_1 + s_2 + s_3$ is odd)

It has been showed in [30] that the $F(3;s_1,s_2,s_3)$ can be presented as:

$$F_o(3;s_1,s_2,s_3) = \rho_1 f_o(3;0,0,0) + \frac{\rho_2}{\pi^2 f_o(3;0,0,0)} + \rho_3. \tag{10}$$

As a result of Eq. (8) and Eq. (10) we express the equivalent two- point capacitance between the sites $(0,0,0)$, and (s_1, s_2, s_3) in the infinite FCC lattice as:

$$C_o(s_1,s_2,s_3) = C \Big/ \left[\delta_1 f_o + \frac{\delta_2}{\pi^2 f_o} + \delta_3\right]. \tag{11}$$

where δ_1, δ_2, and δ_3 are fraction numbers related to ρ_1, ρ_2, and ρ_3 as $\delta_1 = 1 - \rho_1$, $\delta_2 = -\rho_2$, and $\delta_3 = \rho_3$. In [30] one can find values of ρ_1, ρ_2, and ρ_3 (i.e., rational numbers) for different lattice sites. One can obtained further values of ρ_1, ρ_2, and ρ_3 by using Morita's [31] recurrence formulae.

Watson was the first physicists who calculated the lattice GF at the origin of the infinite FCC lattice. He showed that [32]:

$$f_o(3;0,0,0) = F(3;0,0,0) = \frac{\sqrt{3}}{2}[K(k)]^2 = \frac{3\Gamma^6(\frac{1}{3})}{2^{\frac{14}{3}}\pi^4} = 0.4482203944. \qquad (12)$$

where $k = Sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

Table 1 below shows selected values for δ_1, δ_2, and δ_3 at different selected sites.

As the distance between the sites $(0,0,0)$, and (s_1,s_2,s_3) increases to infinity, then the equivalent capacitance approaches a definite value of 2.231045290. This is due to the fact that $\lim_{\alpha\to\infty}\int_a^b \phi(q)Cos\alpha q dq \to 0$ for any integrable function $\phi(x)$. Thus,

$f_o(3;s_1,s_2,s_3) \to 0$, and as a result Eq. (11) becomes

$$\frac{C_o(s_1,s_2,s_3)}{C} \to \frac{1}{f_o(3;0,0,0)} \to 2.231045290. \qquad (13)$$

when any of $s_1,s_2,s_3 \to \infty$.

In this paper we have first collected some values for δ_1, δ_2, and δ_3, and secondly some additional values have been calculated for some lattice sites such as: (6,0,0), (6,2,0), (6,4,0), (7,1,0), (7,3,0), and (8,0,0)using Morita's equation for the lattice GF of the infinite FCC lattice [31].

Table 1: Values for δ_1, δ_2, and δ_3.

The Site (s_1,s_2,s_3)	δ_1	δ_2	δ_3	$C_o(s_1,s_2,s_3)/C$
000	0	0	0	∞
200	4/3	-1	0	2.6912467200430599475206
400	-16/9	16/3	0	2.4463335575805761115528
110	0	0	1/3	3.0000000000000000000
310	8/3	-5	1/3	2.5105001669482611020593
510	-136/9	91/3	1/3	2.3972766936759840820827
211	4/3	2	-2/3	2.6105230182867137430984
411	64/9	-28/3	-2/3	2.4339309445112423270326
220	-8	-6	16/3	2.5558648151981945370945
420	-4/9	-275/3	64/3	2.4232026500144180557675
321	28/3	29	-31/3	2.4656716859523287036237
521	788/9	-95/3	-95/3	2.3854904926276416325342
222	4	-15	2	2.4869497312850815346469
422	-728/9	62/3	32	6.1849189659389392528878
330	-144	-186	107	15.381582560361849327871
530	-688	-2316	2497/3	2.3757257842270813733595
431	2840/9	2101/3	-898/3	5.6440730018413808035644
332	-4	-120	88/3	2.4143275856241278241596
532	-8932/9	-1598/3	1697/3	2.3675307364677861930342
433	1712/9	502/3	-368/3	2.3760193122849702522382

440	-9056/3	-4960	7424/3	2.3807256451766498428721
541	82936/9	56735/3	-8405	2.3605003316502965968666
442	-8212/9	-9059/3	1092	2.3716352424997035455946
543	159056/45	12800/3	-7645/3	2.3491604100694411817216
444	-19024/15	464	464	2.3517680592269268035709
550	-218480/3	-133150	188225/3	2.3491107441278104173665
552	-1596212/45	-249428/3	34694	2.3444460074084493834107
554	-110992/9	-20534/3	21226/3	2.3331396827396659410602
600	1924/75	-49	0	2.3708368105606554889613
620	-56488/225	862/3	48	2.3635018588942120202977
640	-2155436/75	-67681	84544/3	2.3466246151535631148156
710	1425407/9600	-294	1/3	2.3489672765368705648561
730	-170742787/28800	-7201/3	9601/3	2.3403254456564729891338
800	-89212757/352800	73984/147	0	2.3332703350342874075733

4. Results and Discussion

The following points have been discussed and studied concerning the two-point capacitance in an infinite FCC networks composed of capacitors having the same capacitance C. First of all it has been showed in Eq. (11) that the equivalent capacitance between the sites $(0,0,0)$, and (s_1,s_2,s_3) in the infinite FCC network can be represented by π, f_o, and some rational constants (i.e., δ_1, δ_2, and δ_3). Secondly, we calculated some values for the rational constants δ_1, δ_2, and δ_3 which means the net capacitance has been obtained, and some of these values have been presented in Table 1. Thirdly, the equivalent capacitance between the sites $(0,0,0)$, and (s_1,s_2,s_3) in the infinite FCC network has been plotted against the site (s_1,s_2,s_3).

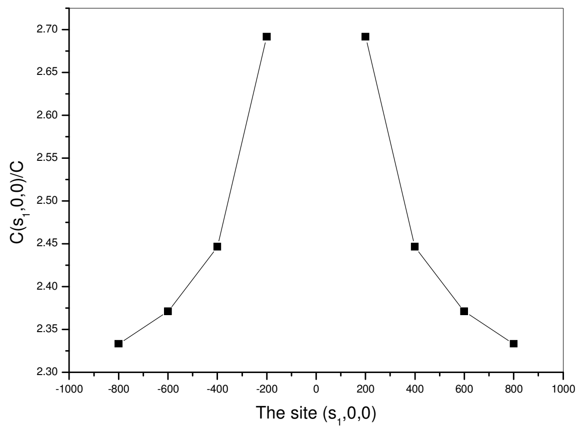

Fig. 1: The calculated capacitance between the origin $(0,0,0)$ and the lattice site (s_1,s_2,s_3) along [100] direction of the infinite FCC lattice

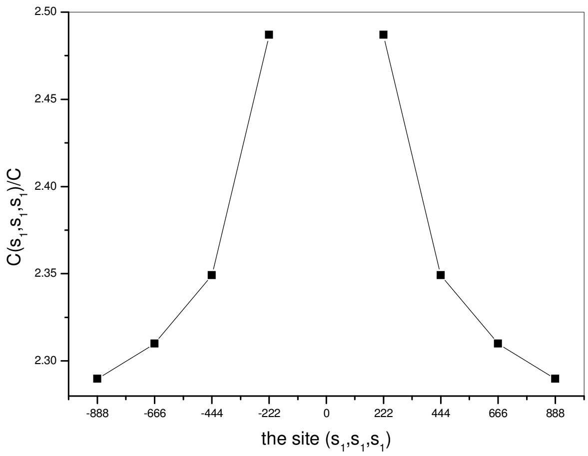

Fig. 2: The calculated capacitance between the origin $(0,0,0)$ and the lattice site (s_1,s_2,s_3) along [111] direction of the infinite FCC lattice

Figure 1 presents the net capacitance along [100] direction, while Fig. 2 presents the net capacitance along [111] direction. It is clear from both Figures 1, and 2 that the net capacitance is symmetric in an infinite FCC lattice along [100] and [111] directions under transformation $s_1,s_2,s_3 \rightarrow -s_1,-s_2,-s_3$, and this is expected due to the inversion symmetry of FCC lattice.

Finally, we discussed the case where the distance the sites $(0,0,0)$, and (s_1,s_2,s_3) approaches infinity. We have showed that in this case it goes to a definite value of 2.231045290.

5. Conclusion

The GF technique was used to study a three- dimensional FCC infinite network of identical capacitors C. The equivalent two-point capacitance between the origin and a lattice point was expressed in terms of f_o and some rational values (δ_1, δ_2, and δ_3) . Some of these values were calculated and the net capacitance was obtained numerically.

The net capacitance along [100] and [111] direction was plotted against site and the calculations and plots showed translational symmetric behavior along these directions and this is due to the fact that the infinite network is perfect and contains no impurities. Finally, it was shown that as the separation between the sites goes to infinity the effective or net capacitance goes to definite value of 2.231045290

Acknowledgment

Authors would like to thank Palestine Technical University- Kadoorie for the financial support.

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