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Assorted Secure and Co - Secure Domination Number of Some Torus, Fractal Networks and Honeycomb Structure Models

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Abstract: - Network on chips with various mesh relations proposed new architect designs on base of 3D formation. Each networks used here are of unique structures, properties, topological indices, different degrees and vary also with its extension view of dimensions. In this paper, we extract the higher degree nodes and placed as defenders to support and secure three times larger of its total number of nodes. Securing high confidential networks with few defenders is a risky factor of carry out the output. The nexus here maneuvered are with wide applications. Circuitry like Honey comb H_C (N), Honeycomb Cup H_CC (N), Honeycomb Cage H_CCA (N), Honeycomb Rhombic Torus HRoT_(N,M), Pyrene PY(N), Pyrene Torus PT(N), Silicate Triangular Fractal ST_F (N) with varieties of properties are used here. Secure, co – secure, strong secure, strong co – secure, perfect secure, perfect co – secure, perfect strong secure, perfect strong co – secure dominating sets of these networks are worked out in this paper with exact minimum cardinality. Some are related to one another and some are equal to one another and such kinds are also resulted here. Replacement of attacked defenders by its neighbor term marked as valuable part by the arrangement of unique node as co – defenders. Handed down in online games, in structural evolution of metallic glasses, to describe silica aero gels, implementation of a spoke hub distribution paradigm in computers, to reduce design complexity for communication scheme, epoxy resins for electrical insulation.

Keywords: communication, dominating, neighbor, structural

1. INTRODUCTION:

A topological index is a molecular descriptor derived from the molecular structure of a chemical compound. Tremendous indices are nowadays used to analyze mathematical values and predict various physical properties of drugs without determining bond lengths and angles. Numerical labeling are assigned to edges and vertices i.e.) atoms and its bonds. When dimensional network are taken out, n level is separated in fact of row wise and atom wise. Securing such high pertinent network is an imperilment work cause to emit and supply the secure related base on Roman domination. Securing network protect confidential information from undiagnosed access, unknown users, disasters from naturally or artificially attackers without any aware. Antivirus software tools to prevent from infected files, permission modification access. Work related to inverter process, when allotted high authorized nodes are defected, then within a fraction of second neighbors of the nodes act according in place and manages to rectify on handling the remaining nodes with low degree. Though suggested defender node is of high tendency, one of the neighbor with second level priority to regulate outputs. Such protection level division convenient nodes with minimum cardinality for interconnection, under wraps networks vary with mesh; torus and rhombic torus are resulted in this paper.

2. MATERIALS AND METHODS

2.1 Perfect strong secure and perfect strong co – secure domination sets:

(i) Let G be a loop less, interconnected, in directed, non – isolated graphs. A subset D of G is a *dominating set* if every vertex $u \in D$ is dominating every$v \in V - D$. The minimum cardinality is denoted by $\gamma(G)$ [9].

(ii) A dominating subset D of V of G is a *strong dominating set* if every $u \in D$ is dominating $v \in V - D$ such that $\deg(u) \geq \deg(v)$. The minimum cardinality is denoted by $\gamma_{sd}(G)$ [15]

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(iii) A subset D of V of G is a *strong secure dominating set* if for each $v \in V-D$ there exists $u \in D$ such that $D\backslash\{u\} \cup \{v\}$ and $\deg u \geq \deg v$ is a dominating set. The minimum cardinality is denoted by $\gamma_{ssd}(G)$ [5]

(iv) A subset D of V of G is a *strong co – secure dominating set* if for every $u \in D$ there exists $v \in V-D$ such that $D\backslash\{u\} \cup \{v\}$ is a dominating set and $\deg u \geq \deg v$. The minimum cardinality is denoted by $\gamma_{scsd}(G)$ [5]

(v) A subset D of V of G is a *perfect strong secure dominating set* if for every vertex in $v \in V-D$ there exists a unique vertex $u \in D$ such that v is adjacent to u and $D\backslash\{u\} \cup \{v\}$ is a dominating set and $\deg u \geq \deg v$. The minimum cardinality is denoted by $\gamma_{pssd}(G)$ [4]

(vi) A subset D of V of G is a *perfect strong co-secure dominating set* if for every vertex in $u \in D$ there exists unique vertex in $v \in V-D$ such that $D\backslash\{u\} \cup \{v\}$ is a dominating set and $\deg u \geq \deg v$. The minimum cardinality is denoted by $\gamma_{pscsd}(G)$ [4]

2.2 Properties of Networks:

2.2.1 HONEYCOMB NETWORK $H_C(N)$:

Honeycomb's tooling enables software developers to debug live software applications especially those using micro service architecture. It accepts telemetry from applications instrumented with the open Telemetry SDKs in addition to structured JSON data on other custom integrations. The nodes design and its interconnection used to predict the bioactivity of chemical compounds. Chiral molecular honeycomb networks can be formed on Au (111). Using this network structure, large amount of suppliers can be protected with a high load of minimum number of security. Making advantage of domination in these kind of Networks defend the neighboring nodes from attacks of unknown accessed users.

$$V(G) = 6N^2; E(G) = 9N^2 - 3N; \Delta(G) = 3; \delta(G) = 2$$

2.2.2 HONEYCOMB CUP NETWORK $H_{CC}(N)$:

This network is the arrangement of n benzenes in addition of n - 1benzenes adjacent on upper and lower of n benzenes. Honeycombs integrated distributed tracing let to do a deep dive into end to end individual user experiences and compare findings across multiple level. Honeycomb cup structure provides a material with minimal density and related high out of plane compression properties. The difference between honeycomb and its cup structure is change in dimension with change in count of vertices. This structure's 2 - rainbow domination number is detailed in [10].

$$V(G) = 2(3N^2 + 4N + 1); E(G) = 9N^2 + 9N + 1; \Delta(G) = 3; \delta(G) = 2$$

2.2.3 HONEYCOMB CAGE NETWORK $H_{CCA}(N)$:

Connections arranged between two Honeycomb networks of equal dimension with only connection of edges from 2 degree vertices of $H_C(1)$ and $H_C(2)$. Row 1 in $H_C(1)$ consists of two vertices with degree 2, which is linked to row 1 in $H_C(2)$ vertices with degree 2. After these connections, the graph assembled here is a planar graph. The hardware cost of honeycomb tore is very expansive as it has more edge crossings. Metric dimensions of this network and distinct representations using p – dimensional vector of distance coordinates are formulated in [14]. The structure of $H_C(N)$ when doubled and connecting using edge between degree 2 nodes $H_C(N)$ *and* $H_C(N)$ forming a cage like structure defined as $H_{CCA}(N)$

Ie) $\Delta(G) = \delta(G) = 3$

2.2.4 SILICATE TRIANGLE FRACTAL NETWORK ($ST_F(N)$):

Based on silicate structure, fractal model is upset in widening the silicate triangle. Fractals are patterns that are self similar iteratively with similar properties. Fusion of metal oxides with sand leads to preparation of silicate materials. Distribution of degree under renormalization or self replicating structure of silicate triangle with leads to fractal network. The graph starts with single silicate. Then enlarging three corners with each carrying single silicate the second level ends with total of 4 silicates. Third stage is addition of three silicates to each three silicates of second level. Therefore $St_0 \subseteq St_1 \subseteq St_2 \subseteq St_3 \subseteq St_4 \ldots. \subseteq St_n$. Microchip production, intercellular products, telecommunication with enormous interlinks are the fact run out using this structure. Unique way of renormalize without relaxing other process of chips and re changing the model by same figure is this model's hope. Infinite number of times the fractal happens. Several results based on this structure is resulted in [8]

$$V(G) = \frac{3(3^{n+1} - 1)}{2} + 1; E(G) = 3(3^{n+1}) - 3; \Delta(G) = 6; \delta(G) = 2$$

2.2.5 PYRENE NETWORK AND PYRENE TORUS NETWORK PY(N) AND PT(N):

In Quantitative Structure – Activity Relationship and Quantitative Structure – Property Relationship, properties like peak wavelength solar cell applications are well – Informed. Physical or chemical properties of the given structure of a molecule demonstrated in this kind of relationship. QSAR predict activities of new chemicals whereas QSPR differentiate behaviors of chemical molecules. Pyrene is a typical polycyclic aromatic hydrocarbon (PAH). It is a valuable molecule probe for fluorescence spectroscopy in sight of its high quantum yield and lifetime. Reversed phase HPLC with fluoric metric detection is a rapid and sensitive method which allows reliable determination of the concentration of traces of benzoic pyrene in edible oils and fats. The domination number and connected domination number of pyrene is in [7] [11]. The fluorescence of Pyrenees is significantly quenched when attached to nano materials suggesting strong contribution from the paramagnetic $Cu^{2+} ions$. Pyrene torus is made by interconnecting its nodes by bond formation between node 1 and node n of degree 2.

$V\big(PY(n)\big) = 2n^2 + 4n;\ E\big(PY(n)\big) = 3n^2 + 4n - 1;\ \Delta\big(PY(n)\big) = 3;$

$\delta\big(PY(n)\big) = 2.$

$V\big(PT(n)\big) = 2n^2 + 4n;\ E\big(PT(n)\big) = 3n^2 + 6n;$

$\Delta\big(PY(n)\big) = 3 = \delta(PY(n))$

The Networks used in upcoming theorems are designed in such a way that its connections are applied to find assorted secure and co – secure domination numbers.

3. MAIN RESULTS AND DISCUSSION:

THEOREM 3.1: The perfect strong co-secure domination number of honeycomb network is $\gamma_{pscsd}\big(H_C(N)\big) = 3N^2 - 2^N + 2$

Proof: Let $H_C(2)$ with 24 vertices where 12 vertices are of degree 3 and 12 vertices is of degree 3. By choosing these 12 vertices in D leads to secure remaining nodes in V – D. Now to find the minimum cardinality and to replace the defected D by neighbors belong to V – D.

Let D = $\{4, 6, 7, 8, 10, 11, 14, 15, 17, 18, 19, 21\}$…… (1)

Now, $D\backslash \{4\} \cup \{1\}\ or\ D\ \{4\} \cup \{2\}$ is not a dominating set, similarly changing these nodes without changing cardinality also leads to non – dominating sets. To support {1, 2}, add {5} to D. Rearranged D named D' has one vertices as follows.

$D' = \{4,5,6,7,13,15,16,18,22,25\}\ ie)\ |D| = 10$

We can verify that, none of the D' vertices when replaced by its higher degree neighbors failed to result the dominating set. Hence $\gamma_{pscsd}\big(H_C(2)\big) = 10$

Even dimension D nodes and odd dimension D nodes are different. For N level dimensional networks, select D by filtering one by one node.

$\frac{V(G)}{2} = \frac{6N^2}{2} = 3N^2$. Directly reducing 2 times $(2^{N-1} - 1)$ vertices from these $3N^2$ results in minimum cardinality of perfect strong co – secure dominating set of $H_C(N)$.

Hence $\gamma_{pscsd}\left(H_C(N)\right) = 3N^2 - 2(2^{N-1} - 1) = 3N^2 - 2^N + 2.$

THEOREM 3.2: The domination number, strong domination number, co - secure domination number and perfect co - secure domination number of $H_{CC}(N)$ are

$$\gamma\left(H_{CC}(N)\right) = \begin{Bmatrix} \left(\frac{3N^2 + 4N + 3}{2}\right) + 1 \ \ if N \ is \ odd \\ \left\lfloor\frac{(3N^2 + 4N + 3)}{2}\right\rfloor \ if \ N \ is \ even \end{Bmatrix}$$

$$\gamma_{sd}\left(H_{CC}(N)\right) = 2N^2 + 4N + 1$$

$$\gamma_{csd}\left(H_{CC}(N)\right) = 2N^2 + 4N$$

$$\gamma_{csd}\left(H_{CC}(N)\right) = \gamma_{pcsd}(H_{CC}(N))$$

Proof:

Random selection of dominating set without including strong degree or weak degree nodes of $H_{CC}(N)$. Let

$D \ of \ H_{CC}(1) = \{1, 7, 9, 11, 16\}$

$D \ of \ H_{CC}(2) = \{1, 6, 8, 18, 20, 22, 24, 26, 35, 37, 39\}$

$D \ of \ H_{CC}(3) = \{5, 7, 8, 10, 21, 23, 25, 28, 30, 32, 40, 48, 50, 52, 55, 57, 59, 67, 70, 73, 76, 78\}$

Continuing these way for both cases of odd and even dimension consequence the domination number as

$$\gamma\left(H_{CC}(N)\right) = \left\lceil\frac{3N^2 + 4N + 3}{2}\right\rceil + 1 \ where \ N \ is \ odd$$

$$\gamma\left(H_{CC}(N)\right) = \left\lfloor\frac{3N^2 + 4N + 3}{2}\right\rfloor where \ N \ is \ even$$

The minimum cardinality of co – secure domination number of $H_{CC}(N) \geq$ The minimum cardinality of domination number of $H_{CC}(N)$.

The nodes of D for $\gamma_{csd}\left(H_{CC}(N)\right)$ are

$D \ of \ H_{CC}(1) = \{1,5,6,11,12,16\}$

$D \ of \ H_{CC}(2) = \{1,5,6,11,13,14,25,27,28,30,35,36,37\}$

$D \ of \ H_{CC}(3) = \{5,6,7,8,10,21,23, \dots., 76,78\}$

Continuing in this way, we get $\gamma_{csd}\left(H_{CC}(N)\right) = 2N^2 + 4N$

Re - modification of D of co – secure domination number with addition of at most one node results in secure dominating set. In co – secure dominating set, at least one vertex is not a defender though it is in dominating set.

The secure dominating set of $H_{CC}(1), H_{CC}(2), .. H_{CC}(N)$ are

$D \ of \ H_{CC}(1) = \{2,5,6,10,11,12,15\}$

$D \ of \ H_{CC}(2) = \{1,2,6,11,12,13,19,22,23,25,26,29,34,35,36,40,41\}$

$\Rightarrow D \ of \ H_{CC}(N) = \{1,2,6,11,12,19, \dots. N-6, N-2, N-1\}$

$\Rightarrow \gamma_{sd}\left(H_{CC}(N)\right) = 2N^2 + 4N + 1$

The uniqueness of D satisfied for both co – secure dominating set and perfect co – secure dominating set are same. But vary in secure and perfect secure dominating set.

Redefining pcsd set varied from csd set,

$D\ of\ H_{CC}(1) = \{1,4,8,9,13,16\}$

$D\ of\ H_{CC}(2) = \{3,4,5,13,14,15,16,22,26,28,29,35,37,38,39,42\}$

$D\ of\ H_{CC}(N) = \{3,4,5,13,14,15,16, \dots N-4, N-3, N\}$

$\Rightarrow \gamma_{csd}\big(H_{CC}(N)\big) = \gamma_{pcsd}(H_{CC}(N))$

THEOREM 3.3: If $\gamma_{pcsd}\big(H_{CC}(N)\big) < 2N^2 + 4N$, then at least one vertex in G is not dominated by vertices in D

Proof: We prove this by considering the subset D of $H_{CC}(2)$.

$D = \{3,4,5,13,14,15,16,22,26,28,29,35,37,38,39,42\}$

$V - D = \{1,2,3,7,8,9,10,11,12,17,18,19,20,21,23,24,25,27,30,31,32,33,34,36,40,41\}$

Suppose $D - \{5\}$, then $\{8\}$ is not dominated by any nodes in D. Also if $\{5\} \in D$ is attacked or repaired, then the unique vertex to replace it is $\{8\} \in V - D$. In general, this kind of selection of D with unique $v \in V - D$ for N^{th} network results $\gamma_{csd}\big(H_{CC}(N)\big) = 2N^2 + 4N = \gamma_{pcsd}\big(H_{CC}(N)\big)$

THEOREM 3.4: For Honeycomb cage network,

(i) $\gamma_{pcsd}\big(H_{CCA}(N)\big) = 4N^2 + (N)2^{N-2} + 2$

(ii) $\gamma_{sd}\ \big(H_{CCA}(N)\big) = \gamma_{csd}\big(H_{CCA}(N)\big) = 4N^2 + (N)2^{N-2}$

Proof of (i):

The network is large, also vertices doubled. $V\big(H_{CCA}(1)\big) = 24 + 24 = 48$. All vertices is of degree 3.The perfect strong co – secure dominating set is

$$D = \{1,3,5,11,12,13,14,20,22,23,27,28,29,34,36,37,40,42,45\}$$

If $u \in D$ is replaced by $v \in V - D$, remaining 2 nodes is dominated by another $u \in D$. It is not necessary of those neighbors to be placed in replacement of u when defected.

Now consider $H_{CCA}(2)$. N (1) = {2, 9, 47}. If {1} is attacked or failed in its process, though $D\backslash\{1\} \cup \{47\}$ is a Dom set, $D\backslash\{1\} \cup \{2\}$ *or* $D\backslash\{1\} \cup \{9\}$ is not a Dom set. Also $D\backslash\{3\} \cup \{43\}$ is Dom set, $D\backslash\{3\} \cup \{9\}$ or $D\backslash\{3\} \cup \{7\}$ is not a Dom set. $D\backslash\{5\} \cup \{6\}$ Is Dom set, $D\backslash\{5\} \cup \{2\}$ *or* $D\backslash\{5\} \cup \{9\}$ is not a Dom set. Row in half of the network has 4 vertices where all nodes are connected. I.e.) cycle graph attained is {3,43,44,40,36,35,39,43,3}. Among these 7 nodes 3 are dominating nodes. Hence on compare to 24 nodes of first $H_C(N)$ 7*3 = 21 nodes have 3*3 = 9 dominating nodes and remaining 3 nodes have 1 dominating nodes. Let the dominating set of first and second $H_C(N)$ are D', D''.

$\Rightarrow |D'| = 10$; $|D''| = 10$. The dominating set of the cage formed $H_C(1)$ that is $H_{CCA}(1)$ with the minimum cardinality is $|D| = |D'| + |D''| = 20$. Some $v \in D$ have the ability of all its neighbors to associate in replacement. But to be noted that almost one is required for co – defenders and other two nodes are co – defenders of some other dominating nodes with uniqueness. Continuing in this way, the minimum cardinality of perfect strong co – secure dominating set for $H_{CCA}(2)$ is $4(2)^2 + 2^{2-2}(2) + 2 = 20$.

Proof of (ii):

The strong secure dominating set of $H_{CC}(2)$ is

$D = \{4,5,7,10,12,13,15,18,20,21,25,28,29,36,37,44,45,48\}$

To find D of $H_{CCA}(3)$. Random selection of 24 dominating nodes from first $H_C(3)$ and 18 dominating nodes from second $H_C(2)$. $V(G) = 54 + 54 = 108$. Ie) $V(G) = 12N^2 = 108$. Number of rows of N is 4N.

Consider R_1 vertices $\{1,2,3, ... N\}$ R_2 vertices $\{N+1, N+2, ... 2N+1\}$ R_3 vertices $\{2N+2, 2N+3, ... 3N+2\}$ R_4 vertices $\{3N+3, 3N+4, ... 4N+4\}$…. R_{4N} vertices $\{6N^2 - (N-1), 6N^2 - N, ... ,6N^2\}$ Edge formed between two Honeycomb Networks major on 2 degree vertices from each row. The connection between $H_C(N)$ and $H_C(N')$ is $H_{CCA}(N)$. The dominating subset D from $R_1, R_3, R_5, R_7, R_9, R_{12}, ...$ taken from $H_C(N)$ and in $H_C(N')$ the dominating subset D from $R_3, R_5, R_8, R_{10}, ...$ are taken. In $H_C(N)$, $R_3, R_5, R_7, R_9, ...$ in $\frac{4N}{2}$ rows and $R_{2N+1}, R_{2N+2}, ...$ from remaining $\frac{4N}{2}$ rows are all not connected to second $H_C(N')$ .Also these are nodes of degree 3 interconnected in same $H_C(N)$. Therefore selecting these nodes in D from these rows are mandatory. Similarly in $H_C(N')$, these nodes are in D. Hence number of nodes from $R_3, R_5, R_7, ...$ and R_{2N+1}(2 end vertices) and number of nodes from $R_{2N+2}, R_{2N+3}, ... R_{2N+6} = R_{4N}$ are all in D. In $H_C(N')$, let define rows by R'. Now dominating nodes in R' are $R_3, R_5, ... R_{2N-1}, R_{2N+2}, R_{2N+4}, ... R_{4N-2}$

No. of vertices from R collected to D of $H_C(N)$ is $|D'| = 4N^2$

No. of vertices from R' collected to D of $H_C\ (N')$ is $|D''| = \frac{(2^N N)}{2^2}$

Hence $|D| = |D'| + |D''| = 4N^2 + \frac{2^N N}{2^2}$

OBSERVATION 3.5: The dominating set are independent or connected, but to check the minimum cardinality the lowest number of nodes can be taken and to satisfy the dominating set of all the replacement of D by some V – D.

THEOREM 3.6: The assorted secure and co-secure domination number of silicate triangle fractal network $ST_F(N)$ are

(i) $\gamma(ST_F(N)) = 3^N \left[1 + \frac{1}{3^3}\right] for\ N \geq 3$

(ii) $\gamma_{ssd}(ST_F(1)) = 4$

(iii) $\gamma_{ssd}(ST_F(N)) = 3^N + 3^{N-2}\ for\ N = 2, 3$

(iv) $\gamma_{ssd}(ST_F(N)) = 3^N + 3^{N-2} + 3^{N-4}\ for\ N \geq 4$

(v) $\gamma_{ssd}(ST_F(N)) = \gamma_{scsd}(ST_F(N))$

Proof: The domination number $|D|$ for $ST_F(0)\ is\ 1, ST_F(1)\ is\ 1, ST_F(2)\ is\ 3$.

The triangle structure with centre vertex increases 3 times to next dimension. No. of triangles in $ST_F(2) = 1 + 3; ST_F(3) = 1 + 3 + 3^2; ST_F(4) = 1 + 3 + 3^2 + 3^3; ST_F(5) = 1 + 3 + 3^2 + 3^3 + 3^4$.

Hence the domination number of $N \geq 3$ is

$\gamma(ST_F(3)) = 1 + 3(9),\ \gamma(ST_F(4)) = 3(1 + 3(9)) ...\ \gamma(ST_F(N)) = 3^{N-3}(1 + 3(9))$

$\gamma(ST_F(N)) = 3^N(1 + \frac{1}{3^3})$

Now to find secure domination number with minimum cardinality assume $\gamma(ST_F(N)) + 1 = \gamma_{sd}(ST_F(N))$ where N = 1, 2.

I.e.) $\gamma(ST_F(1)) + 1 = \gamma_{sd}(ST_F(1)) = 2;\ \gamma(ST_F(2)) + 1 = \gamma_{sd}(ST_F(2)) = 4$

On the contrary, let $\gamma(ST_F(2)) < 4$. Let the subset D vertices of $ST_F(1)$ are {1,5,9}.

Now $D \setminus \{1\} \cup \{2\}$ or $D \setminus \{1\} \cup \{3\}\ or\ D \setminus \{1\} \cup \{4\}$ is a dominating set. Also, $D \setminus \{5\} \cup \{6\}$ or $D \setminus \{5\} \cup \{7\}$ or $D \setminus \{5\} \cup \{8\}$ is a dominating set. $D \setminus \{9\} \cup \{10\}$ or $D \setminus \{9\} \cup \{11\}$ or $D \setminus \{9\} \cup \{12\}$ is a dominating set. Note that the number of vertices in $ST_F(1)$ is 13 where {13} is the centre vertex is not cast – off in replacement of any $u \in D$. Therefore rearranging dominating nodes of secure set, we get $D = \{1,5,9,13\}$. $|D| = 4 = \gamma_{sd}(ST_F(2))$

Let N = 3. To prove $\gamma_{sd}\big(ST_F(3)\big) = 10$

We initiate the arrangement of security from the foremost center node {1}. Let N denote neighbors of the corresponding node.

$N(1) = \{2,3,4\}; N(2) = \{5,6,7\}; N(3) = \{8,9,10\}; N(4) = \{11,12,13\};$

$N(5) = \{14,15,16\}; N(6) = \{17,18,19\}; N(7) = \{20,21,22\};$

$N(8) = \{23,24,25\};\ N(9) = \{26,27,28\};\ N(10) = \{29,30,31\};$

$N(11) = \{32,33,34\}; N(12) = \{35,36,37\}; N(13) = \{38,39,40\}$

Deg of {2,3,4} =Deg of {5,6,7,8,9,10,11,12,13}

Deg of {1} = Deg of {14, 15, 16, 17, 18, 38, 39, 40}

Choose greater degree nodes to minimize the cardinality and maximize neighbor support. Therefore $D = \{5,6,7,8,9,10,11,12,13\}$ when replaced by its neighbors $V - D = \{14,15,16, \dots ,38,39,40\}$ are dominating sets. $|D| = 9$. Remaining nodes are {1,2,3,4} where {2,3,4} are of higher degree and degree of {1} is 3.

$D\backslash\{1\} \cup \{2\}$ is Dom set, $D\backslash\{1\} \cup \{3\}$ is Dom set, $D\backslash\{1\} \cup 4$ is Dom set and $\deg(u) \geq \deg(v)$. Therefore $D = \{5,6,7,8,9,10,11,12,13\} \cup \{1\} \Rightarrow |D| = 10$

Let $N = 3$. Suppose $\gamma = \gamma_{ssd}$, then $\gamma = \gamma_{ssd} = 28$.

This is impossible, since in D the node {1} dominate {2,3,4}. On the other hand, $D = \{1\} \cup$ {outer most triangle silicate center nodes}. But {1} when replaced by $\{2\}\ or\ \{3\}\ or\ \{4\}$ is not a dominating set.

Therefore $\gamma \neq \gamma_{ssd}$. Hence for strong secure dominating set, $D = \{2,3,4\} \cup$ {outer most triangle silicate nodes}

$\Rightarrow |D| = 3 + 27 = 30 \Rightarrow \gamma_{ssd}(ST_F(3))$

Let $N \geq 4$. The collection of dominating nodes in D is centre vertex of N^{th} fractal triangle with union of $(N-2)^{th}$ triangle outer 3 vertices + $(N-4)^{th}$ triangle inner centre vertices + … + inner most fractal triangle centre node. $For\ N = 4, |D| = 3(3^3) + 3(3^1) + 1; For\ N = 5, |D| = 3(3^4) + 3(3^2) + 3^1$

Generalizing these steps results in finding the minimum cardinality of strong secure domination number of $(ST_F(N))$.

Ie) $\gamma_{ssd}\big(ST_F(N)\big) = 3^N + 3^{N-2} + 3^{N-4}\ for\ N \geq 4$

Also $D - \{u\}$ is not a Dom set and each vertex with degree 6 can be replaced by at least one vertex of degree 3. This structure is of C_4 and centre is on D.

$D - \{u\} \Rightarrow |N(u)| = 3$ is not dominated by any other $v \in D$

Ie) $\gamma_{scsd}\big(ST_F(N)\big) = 3^N + 3^{N-2} + 3^{N-4}\ for\ N \geq 4$

RESULT 3.7:

Though $H_C(1),\ H_{CC}(1),\ PY(1),\ H_{CCA}(1)$ vary with number of vertices, initial level of securing sets remains same. That is,

$$\gamma_{csd} = \gamma_{sd} = 3 = \gamma_{pcsd} = \gamma_{pssd}$$

EXAMPLE 3.8: $\gamma_{pcsd} = 6\ and\ \gamma_{psd} = 7\ for\ PY(2)$

Solution: The subset D of Pyrene Network of N = 2 is $D = \{1,4,8,9,13,16\}; V - D = \{2,3,5,6,7,10,11,12,14,15\}$. First to prove, $\gamma_{pcsd} = 6$

$V - D$ nodes are in replacement process, when any of $u \in D$ is defected. $\{2,6,11,15\}$ are dominated by nodes in D. Also, $D\backslash\{u\} \cup \{2\} or \{6\} or \{11\} or \{15\}$ are dominating sets. Consider another dominating set $D = \{1,4,5,11,12,16\}, V - D = \{2,3,6,7,8,9,10,13,14,15\}$

$D\backslash\{u\} \cup \{v\}$ where $u \in D, v \in V - D$ are dominating sets with uniqueness property satisfied. Also $\{6,8,9,10,14\}$ are not in secure process but are dominated. From these results, at least 4 vertices are not satisfying $D\backslash\{u\} \cup \{v\}$

$$\Rightarrow |D| = \gamma_{pcsd}(PY(2)) = 6$$

Now to prove $\gamma_{psd}\big(PY(2)\big) = 7$

Let $D = \{1,4,5,11,12,13,16\}, V - D = \{2,3,6,7,8,9,10,14,15\}$. Each D has one V - D to change when defection occurs.

$|V(G)| - |V - D| = 2$ where these 2 nodes are neighbors of {1} and {16}.

To Prove: Uniqueness

$D\backslash\{1\} \cup \{2\} or \{3\}$ is dominating but $D\backslash\{5\} \cup \{3\}$ is only dominating set. Hence $D\backslash\{1\} \cup \{2\}$ shows the uniqueness. Similarly, $D\backslash\{16\} \cup \{14\} or \{15\}$ is dominating but $D\backslash\{12\} \cup \{14\}$ is only dominating set. Therefore D we minimized from V (G) is a perfect secure dominating set and thus$\gamma_{psd}\big(PY(2)\big) = 7$.

THEOREM 3.9: For $N \geq 2$, $\gamma_{pcsd}\big(PY(N)\big) = N^2 + 2$; $\gamma_{pssd}\big(PY(N)\big) = N^2 + 3$

Proof : N – Number of rings in centre of graph. The extension is from Center to upper level as N $N \to N - 1 \to N - 2 \to N - 3 \to \cdots \to 3 \to 2 \to 1$ and similarly down level as the same from$N \to N + 1 \to N + 3 \to N + 4 \to \cdots 2N - 1 \to 2N$. Each benzene is of C_6 implies atleast 2 from each rings are in D. connected nodes between two rings reduces the number of nodes in D from 4 to 3. Calculating in this manner, domination number from row 1 to row N – 1 and domination number from row N + 1 to 2N are equal.

For $N = 3, |D| = 11; N = 4, |D| = 18$

I.e.) Place less than half of the vertices in D. Each vertex in D can be replaced by at least 2 nodes in V – D.

$\Rightarrow |D| = \frac{V(G)}{2N} = \frac{2N(N^2+2)}{2N} = N^2 + 2 = \gamma_{pcsd}(PY(N))$

Proof of $\gamma_{pssd}(PY(N))$ is similar as above.

Hence$\gamma_{pssd}\big(PY(N)\big) = \gamma_{pcsd}\big(PY(N)\big) + 1 = N^2 + 3$

THEOREM 3.10: [11] If G is a pyrene torus Network, then $\gamma(G) = \frac{2N^2+4N}{4}$

THEOREM 3.11:

$$\gamma_{pcsd}\big(PY(N)\big) = \gamma_{pcsd}\big(PT(N)\big)$$

$$\gamma_{psd}\big(PY(N)\big) = \gamma_{psd}(PT(N))$$

NOTE: The vertices in D of $\gamma_{pcsd}(PY(N))$ and $\gamma_{pcsd}(PT(N))$ are different. But the cardinality of pyrene and pyrene torus are equal. This statement is also true for perfect secure dominating set of$PY(N), PT(N)$. In $PT(N)$, the graph is a regular graph and so deg (u) = deg (v) where $u \in D, v \in V - D.$ Therefore for both $PY(N), PT(N)$

$$\gamma_{pcsd} = \gamma_{pscsd};\ \gamma_{psd} = \gamma_{pssd}$$

THEOREM 3.12: The secure domination number and perfect secure domination number of Honeycomb Rhombic Torus Network is

$$\gamma_{sd}(HRoT_{N,M}) = \frac{MN}{2}\ where\ \ N \geq 2, M = 2N$$

Proof:

We prove this by two cases,

(i) Fixing M and changing N and
(ii) Fixing N and changing M.

Case (i): Fix M and Change N Case (ii): Fix M and Change N

$HRoT_{N,4}$	γ_{sd}
2,4	4
3,4	6
4,4	8
5,4	10

$HRoT_{2,M}$	γ_{sd}
2,4	4
2,6	6
2,8	8
2,10	10

$HRoT_{N,6}$	γ_{sd}
2,6	6
3,6	9
4,6	12
5,6	15
$HRoT_{N,8}$	γ_{sd}
2,8	8
3,8	12
4,8	16
5,8	20
$HRoT_{N,10}$	γ_{sd}
2,10	10
3,10	15
4,10	20
5,10	25

$HRoT_{3,M}$	γ_{sd}
3,4	6
3,6	9
3,8	12
3,10	15
$HRoT_{4,M}$	γ_{sd}
4,4	8
4,6	12
4,8	16
4,10	20
$HRoT_{5,M}$	γ_{sd}
5,4	10
5,6	15
5,8	20
5,10	25

Here we list out $N = 2,3,4,5; M = 4,6,8,10$. Comparing in these manners, we generalize the $N \geq 2$ terms as

$$\gamma_{sd}\left(HRoT_{N,4}\right) = 2N;\ \gamma_{sd}\left(HRoT_{2,M}\right) = \frac{2M}{2}$$

$$\gamma_{sd}\left(HRoT_{N,6}\right) = 3N;\ \gamma_{sd}\left(HRoT_{3,M}\right) = \frac{3M}{2}$$

$$\gamma_{sd}\left(HRoT_{N,8}\right) = 4N;\ \gamma_{sd}\left(HRoT_{4,M}\right) = \frac{4M}{2}$$

$$\gamma_{sd}\left(HRoT_{N,10}\right) = 5N; \gamma_{sd}\left(HRoT_{5,M}\right) = \frac{5M}{2}$$

Hence $\gamma_{sd}\left(HRoT_{N,M}\right) = \frac{MN}{2}$ is proved.

THEOREM 3.13: $\gamma_{sd}\left(HRoT_{2N,M}\right) = \gamma_{sd}\left(HRoT_{N,2M}\right)\ where\ N \geq 2, M = 2N$

OBSERVATION 3.14: The connected nodes of $HRoT_{2,4}$ is equal to the structure of cube with 8 vertices having $\gamma_{sd} = 4$.

Figures of above mentioned Networks with vary in dimension and vertices are given as follows.

Fig 1 $H_C(1) \to H_C(2)$

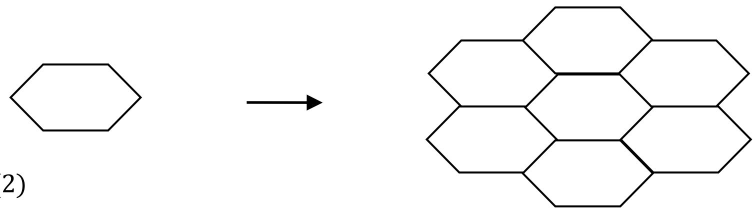


Fig 2 $H_{CC}(1) \to H_{CC}(2)$

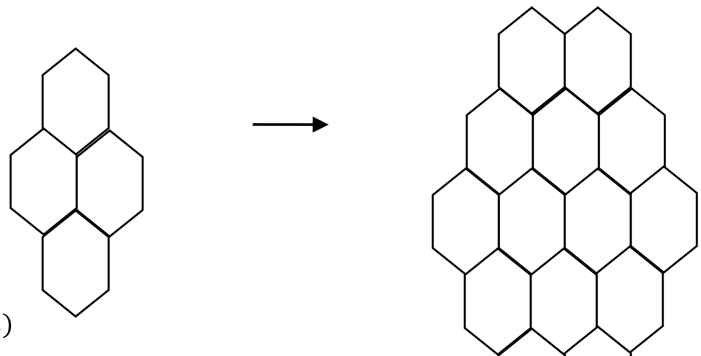


Fig 3$H_{CCA}(1) \to H_{CCA}(2)$

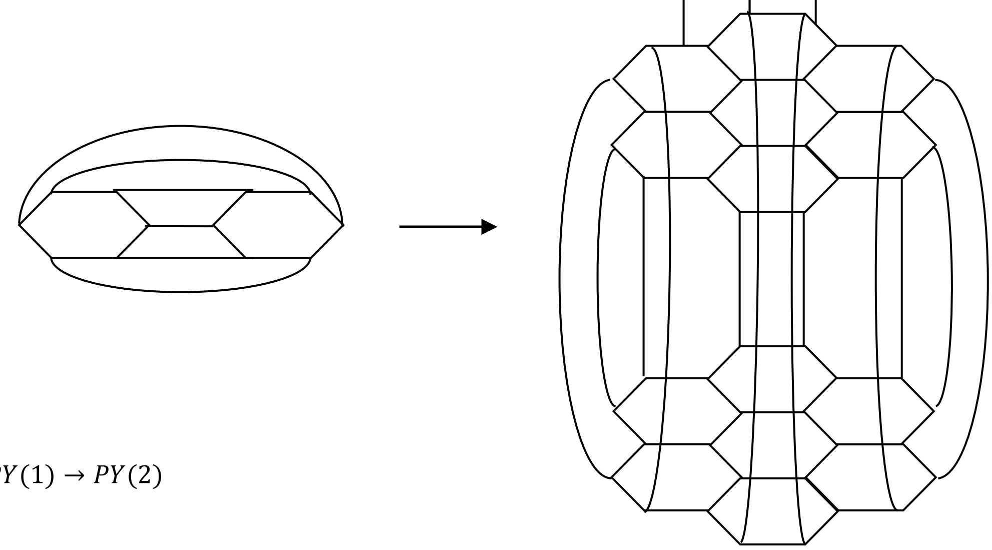


Fig 4 $PY(1) \to PY(2)$

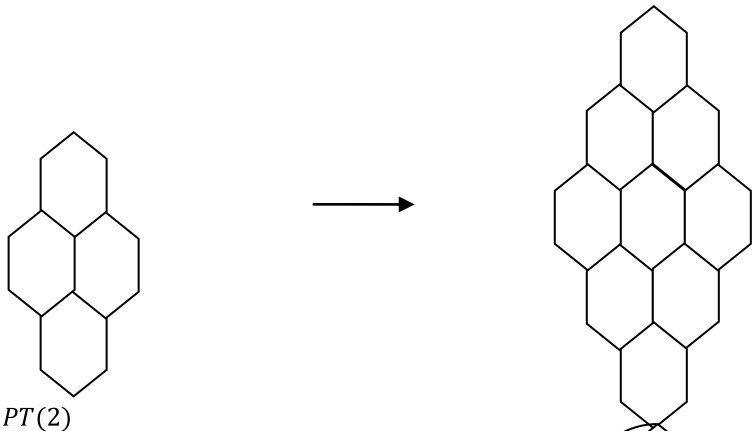


Fig 5 $PT(1) \rightarrow PT(2)$

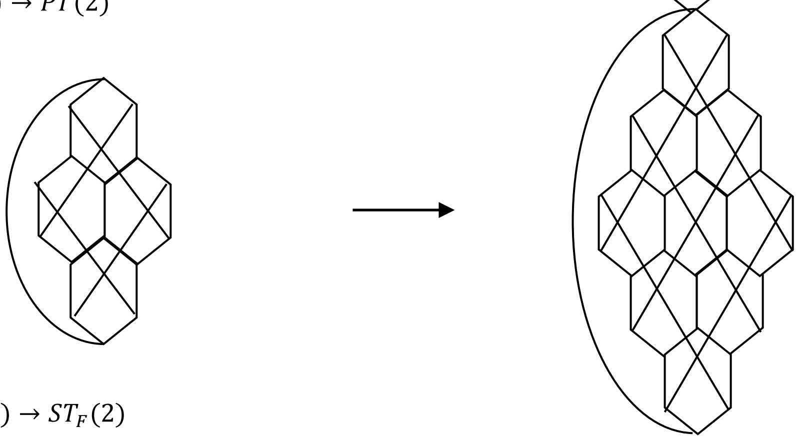


Fig 6 $ST_F(1) \rightarrow ST_F(2)$

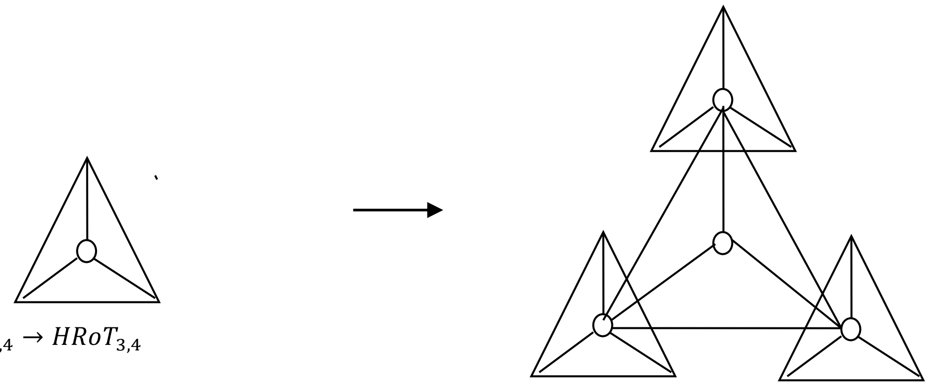
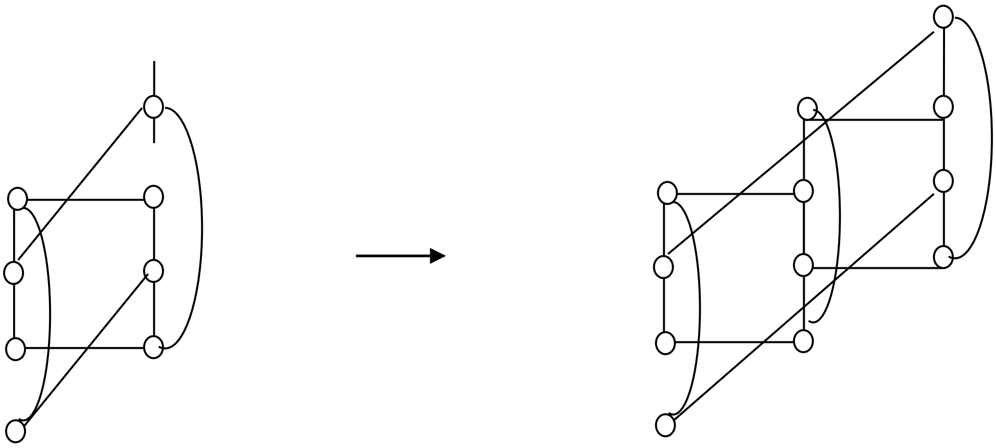


Fig 7 $HRoT_{2,4} \rightarrow HRoT_{3,4}$



CONCLUSION:

Networks application in our day – to – day life is relevant and its designs are highlighted in assorted dominating set of graph theory. In this paper, our aim is to work out the protection of enormous networks with minimum security level. The problems of finding the domination number, secure and co - secure domination number and perfect secure and co – secure domination number of some fused networks are under investigation.

Author contributions:

All authors contributed to the study's conception and workouts. All authors read and approved the manuscript.

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The Authors have no relevant financial or non-financial interest.

Data Availability statement:

Data availability is not applicable to this article.

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