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On Some Properties of Multistep Second Derivative Methods with Constant Coefficients

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Abstract: - Construction and application of the Multistep Second derivative methods usually is connected with the name of Ştörmer, who was the first to construct such methods. This method in one version is fully investigated by Dahlquist. Since the beginning of 20 th century, new areas of mathematics have emerged, the research of which was formulated using Ordinary Differential Equations, ones with a special structure. New modifications have appeared in Multistep Second derivative Methods with constant coefficients. This method is called the Ştörmer or Methods with special structure. Recently have constructed similar methods using other ways, such as using the Spline function used in the power series. Here, for this purpose, the Multistep Second derivative method with constant coefficients is used. All the methods with Second derivative have compared, have given some recommendation for their application, to solving some initial-value problems. Constructed concrete methods some of them have applied to solve model problems. Note that, have defined the condition, which usually have imposed on the coefficients of the Multistep Second derivative methods.

Keywords: Initial-value problem, Ordinary Differential Equation (ODE) of second order, Multistep Second derivative method, Stability and degree, ODE of special structure

I. INTRODUCTION

Ordinary Differential Equations of the second order have been studied since from the age of Newton. However, even now there will be numerous works dedicated to the study of the indicated Ordinary Differential Equation of the second order. Among them, a significant place is occupied by research of Ordinary Differential Equations of the special structure (see for example [1]-[19]).

Let us a consider the following problem:

$$y''(x) = F(x, y(x), \delta y'(x)),\ \ y(x_0) = y_0,\ y'(x_0) = y'_0,\ x_0 \le x \le X. \tag{1}$$

Here, the constant δ - receive the values 0 and 1. It is to say that δ =1 or $\delta = 0$.

Suppose that the continuous solution y(x) of the problem (1) has defined on the segment [x_0, X] and has been the continuous derivatives up to p+1, inclusively. The continuous to totality of arguments function $F(x,s,y)$ has been defined on some close area in which, that has the partial derivatives up to p, inclusively.

As was noted above here, was considered investigation of the numerical solution of problem (1). Therefore let us suppose that the unique continuous solution of problem (1) has defined on some interval [x_0, X]. Let us denoted by the $y(x_i)$ exact value of the solution of problem (1) at the point x_i, but corresponding approximately values by the y_i,here $x_{i+1} = x_i + h$ $(i = 0,1,..,N-1)$ are mesh-points. The h>0 is the step size. Also, suppose that the continuous to totality of arguments function $F(x,y,z)$ has been defined in some closed set in which the function $F(x,y,z)$ has the partial derivatives up to p.

Numerical method for solving problem (1) in one version can be presented as follows (see for example [2]-[10]):

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i F_{n+i};\ \ n = 0,1,2,..,N-k;\ \ \alpha_k \ne 0. \tag{2}$$

The conception of stability and degree for the method of (2), can be presented as following:

Definition 1. The method (2) is called as the stable, if the roots of the following polynomials: $\rho(\lambda) \equiv \alpha_k \lambda^k + \alpha_{k-1}\lambda^{k-1} + \ldots + \alpha_1 \lambda + \alpha_0$ located in the unite circle on the boundary of which there is not multiply root and $|\beta_k| + |\beta_{k-1}| + \ldots + |\beta_1| + |\beta_0| \ne 0$.

Definition 2. The integer value p is called as the degree of the method (2), if the following holds:

$$\sum_{i=0}^{k} (\alpha_i y(x+ih) - h\beta_i y'(x+ih) - h^2 \gamma_i y''(x+ih)) = O(h^{p+1}),\ h \to 0. \tag{3}$$

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Note that in the existing definition for stability of method (2) the condition $|\beta_k|+...+|\beta_0|\neq 0$ is not involved. The condition $\beta_i = 0\ \ (i = 0,1,2,..,k)$ limitation on the coefficients relation with that in this case the method (3) cannot be stable. Because for the convergence of the method (2), the following condition is necessary:

$$\rho(1) = \rho'(1) = 0.$$

Consequently, it follows from here that the $\lambda = 1$is the double root. Therefore in these cases the following definition of stability is usually used:

Definition 3. Method (2) in the case $\beta_i = 0\ \ (i = 0,1,2,..,k)$, called as the stable if the roots of the polynomial $\rho(\lambda)$ located in the unit circle on the boundary of which there is not multiple root except for the double root $\lambda = 1$.

And now let us compare methods receiving from the method (2) and methods receiving from the method (2) in the case $\beta_i = 0\ \ (i = 0,1,2,..,k)$.

§1. On some comparison of the Multistep Methods.

Let us consider some partial case of method (2):

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+i}\,,\ n = 0,1,..,N-k. \tag{4}$$

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h^2\sum_{i=0}^{k}\gamma_i F_{n+i}\,,\ n = 0,1,..,N-k. \tag{5}$$

Note that method (4) is usually applied to solve the following problem:

$$y' = f(x,y),\ y(x_0) = y_0,\ x_0 \le x \le X. \tag{6}$$

The method (4) has been well studied by many authors starts from the 1952 y. (see for example[16]-[32]).

As is known, in the investigation of Multistep Methods with constant coefficients arises question about convergence of these methods, therefore, let us begin from the natural conditions that imposed on the coefficients of these methods (see for example [1]):

A. The coefficients $\alpha_i, \beta_i\ (i = 0,1,..,k)$ are some real number and $\alpha_k \neq 0.$

B. The following characteristic polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k}\alpha_i\lambda^i;\ \delta(\lambda) \equiv \sum_{i=0}^{k}\beta_i\lambda^i$$

don't have common factor different from constant

C. The conditions $\delta(1) \neq 0,\ p \ge 1$ take place.

It follows from here that, $\rho(1) = 0$ is necessary condition for convergence of method (4).

As was noted above for the comparison of the methods of type (2), are used the conceptions of stability and degree. Bakhalov prove that if method of (4) is stable, has the degree of p and $\beta_k = 0$, then $p \le k$ for the k ≤ 10. Dahlquist fully investigated method (4) and prove the following theorem:

Theorem 1. Let the coefficients of method (4) satisfies the condition A, B,C and method (4) has the degree of p, then $p \le 2k$ and method with the degree of p=2k is unit. If method of (4) stable and has degree of p, then $p \le 2[k/2]+2$ in the case $\beta_k \neq 0$ and $p \le k$ for the case $\beta_k = 0$. There are methods with the degree $p_{\max}$ for all the values of k.

If one uses Definition 3 in the investigation of class methods (5), then receives that one can be use the statement of the Theorem 1 in the application of the method (5) to solve some applied problems.

For the comparison above given methods, let us consider method (2). Noted that method (2) can be applied to solve problem (1) and the (6):

In the application of method (2) to solve problem (1), receive:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i f_{n+i} + h^2\sum_{i=0}^{k}\gamma_i y''_{n+i};\ n = 0,1,..,N-k. \tag{7}$$

The values of y''_m -can be calculated by using problem (6). In this case receives: $y''(x) = f'_x(x,y) + f'_y(x,y)f(x,y).$

By denoting $g(x,y) = f'_x(x,y) + f'_y(x,y)f(x,y),$ receive the following:

$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h^2\sum_{i=0}^{k}\beta_i f_{n+i} + h^2\sum_{i=0}^{k}\gamma_i g_{n+i}. \tag{8}$$

Note that one can constructed method of type (8) by using the following method (see for example [31]-[47]):

$$y_{n+sh} = y_n + shy'_n + \frac{(sh)^2}{2!} y''_n + \dots; \; s = 1,2,\dots,k, \tag{9}$$

which is called as the Taylor series.

Let us consider estimation the exactness of method (8). For this aim, let us consider the following theorem.

Theorem 2 (Dahlquist). Let method (8) has the degree of p and stable, then $p \le 2k+2$, if $\beta_k \ne 0, \gamma_k \ne 0$ and $p \le 2k$ if $\beta_k = \gamma_k = 0$. If method (8) is instable, then $p \le 3k+1$. By comparison the results of Theorem 1 and Theorem 2, receive that the method (8) is more promising. But in the application of method (8) to solve problem (6) arises some difficulties which related with the calculation of values $g_m (m \ge 0)$. In the application of method (8) to solve problem (1) does not arise any difficulties.

It is known that in solving applied problems arises question about the construction of more accurate methods. Therefore, scientists tried to construct more accurate methods for solving problem (6). V.Ibrahimov for this aim constructed and investigates the following method:

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i}, \; n = 0,1,..,N-k; \; m > 0, \; \alpha_{k-m} \ne 0. \tag{10}$$

This method is fully investigated by V.Ibrahimov and is proved the following theorem.

Theorem 3(V.Ibrahimov). If method (10) is stable and has the degree of p, then in the class of (10),there are methods with the degree

$$p \le k + m + 1 \; (k \ge 3m)$$

Method (10) by V.Ibrahimovis the called as the forward-jumping, but some authors have called methods of type (10) as advanced.

Let us write method (8) as follows:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{l} \beta_i f_{n+i} + h^2 \sum_{i=0}^{s} \gamma_i g_{n+i}, \; n = 0,1,..,N-k; \; \alpha_k \ne 0. \tag{11}$$

It is obvious that the method (8) can be receive as the partial case from the method (11) in the case s=l=k (see for example [48]-[57]).

Note that maximum value for the degree of the method (11) one can be found by using the results of the following theorem (suppose that $|\beta_0| + |\beta_1| ... + |\beta_l| \ne 0$).

Theorem 4 (V.Ibrahimov). Let method of (11) be stable and has degree of p, $\alpha_k \ne 0$ and $k \ge \max(l,s)$. In this case in the class of method (11) there are stable methods with the degree $p \le l + s$ if k>max(l,s), and the degree $p \le l + s + 2$ if l=s=k. If k<l, then there are stable methods with the degree $p \le l+s+m+2$ for k=2r and $k \ge 3m \; (m = l - k)$,

Bellow will give some examples, regarding these theorems, which confirm the obtained results. Let us note that stable advanced method Störmer type can be presented as follows:

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h^2 \sum_{i=0}^{k} \gamma_i y''_{n+i}, \; n = 0,1,..,k; \; m > 0. \tag{12}$$

Note that presentation of Multistep Methods in the form (12) is more general than the presentation in the form of (5). For the verification, let us look at the following representation of the problem:

$$y''(x) = f(x) + \int_{x_0}^{x} K(x,s,y(s),y'(s))ds, \; y(x_0) = y_0, \; y'(x_0) = y'_0, \; x_0 \le x \le s \le X. \tag{13}$$

Method (12) in simple form can be applied to solve the problem (13).

The definition for stability and degree for the advanced methods has been defined by the same way, which has used for the conception of stability and degree for method (5). For the finding the maximum value of the degree of method (12) , can be used the following theorem:

Theorem 5 (V.Ibrahimov). Let us assume that method (12) is stable and has the degree of p, then the following take place:

$$p \le k + m + 1.$$

And now, let us consider some specific methods that special case of the above studied class of numerical methods. For this, let us consider some partial cases of the method (7), which can be presented as following:

$$y_k = y_{k-1} + hf_{k-1} + h^2 g_{k-1}/2;\ R_n = h^3 y'''/6,\ p=2. \quad (14)$$

$$y_k = y_{k-1} + h(f_k + f_{k-1})/2 + h^2(-g_k + g_{k-1})/12;\ R_n = h^5 y^{(5)}/720,\ \ p = 4, k = 1 \quad (15)$$

$$y_k = (y_{k-1} + y_{k-2})/2 + h(-f_{k-1} + 7f_{k-2})/2 + h^2(11g_{k-1} + 2g_{k-2})/8;\ R_n = 7h^5 y^{(5)}/160;\ p = 4, k = 2. \quad (16)$$

$$y_k = y_{k-1} + h(-f_k + 3f_{k-2})/2 + h^2(17g_{k-1} + 7g_{k-2})/12;\ R_n = 315hy^{(5)}/720;\ p=4. \quad (17)$$

$$y_k = (y_{k-1} + y_{k-2})/2 + h(5f_k + 16f_{k-1} + 3f_{k-2}) + h^2 g_{k-1}/8;\ R_n = -h^5 y^{(5)}/120;\ p=4. \quad (18)$$

$$y_k = y_{k-2} + 2h(32f_{k-1} - 73f_{k-2} + 56f_{k-3})/15 + 2h^2(7g_{k-2} + 6g_{k-3})/5,\ R_n = 7h^6 y^{(6)}/450,\ p = 5. \quad (19)$$

$$y_k = y_{k-2} + 2h(2f_{k-1} + 8f_{k-2} + 5f_{k-3})/15 + 2h^2(2g_{k-2} + g_{k-3})/5,\ R_n = -h^6 y^{(6)}/900,\ p = 5. \quad (20)$$

$$y_k = y_{k-1} + h(101f_k + 128f_{k-1} + 11f_{k-2})/240 + h^2(-13g_k + 40g_{k-1} + 3g_{k-2})/240;\quad R_n = O(h^7),\ p = 6. \quad (21)$$

$$y_k = (13y_{k-1} - y_{k-2})/12 + h(-8939f_{k+1} + 184599f_k + 168507f_{k-1} - 11527f_{k-2})/362880 + h^2(1241g_{k+1} - 7389g_k + 12753g_{k-1} - 501g_{k-2})/362880;\ R_n = 10^{-5}h^9 y^9,\ p = 8. \quad (22)$$

Let us note that methods (14) and (15) have been received from the method (7) for the k=1, therefore, they can be taking as the one-step methods. Note that these methods have the degree p=2 and p=4. Methods (16) - (18) are received from the method (7) in the case k=2,and have the degree p=4.Local traction error for the methods (16) , (18) and (17), (18) and also method (19),(20), have different signs, so it can be assumed that the exact meaning is between of them.Noted that the coefficients in the asymptotic presentation of the local truncation errors for these methods (19), (20) also have different signs. So, these methods can be accepted as the bilateral (two-sided) method. Noted that method (21) is implicit, but method (22) is the Advanced (forward jumping) Multistep Second Derivative Methods and have the degree p=6 and p=8 respectively. Here, have been proposed some ways to construct stable Multistep Methods (explicit, implicit and advanced type) for solving problems of (1), (6) and (13) by using the some properties of this problem.

Numerical Results

For the illustration of the results receiving here, let us applied following Störmer-Verletmethod and the method with fractional step-size:

$$\begin{cases} y_{n+2} = 2y_{n+1} - y_n + h^2(y''_{n+2} + 10y''_{n+1} + y''_n)/12, & (23) \\ y_{n+2} = 2y_{n+1} - y_n + h^2(5y''_{n+1-\gamma} + 14y''_{n+1} + 5y_{n+1+\gamma})/24,\ \gamma = \sqrt{10}/5. & (24) \end{cases}$$

to solve following problem:

$$y'' = (2 + 4x\lambda)\exp(\lambda x) + \lambda^2 y(x),\ y(0) = 0,\ 0 \le x \le 1, \quad (25)$$

the exact solution can be presented as following: $y(x) = x^2 \exp(\lambda x)$.

A well-known method is chosen here: the Störmer method (with step size h) and method Störmer 1 (with the step size h/2). These methods have different properties. Let us note that the solution of the problem (25) also has different properties. For example, if the argument x increases according to the solution, it also increases for the λ > 0, but for the λ < 0 the solution will decrease. Taking into account this, here have considered the cases when λ and h get different values.

The receiving results have tabulated in the table 1 and table 2.

Table1.Results for h=0.1 and λ =1.

x	***Method Störmer***	***Method 24***	***Method Störmer 1***
0.2	***1.4E-07***	***8.0E-12***	***3.1E-09***
0.4	***9.5E-07***	***5.3E-11***	***2.0E-08***
0.6	***2.6E-06***	***1.4E-10***	***5.4E-08***

0.8	*5.5E-06*	*3.0E-0*	*1.1E-07*
1.0	*9.8-E06*	*5.3E-10*	*2.0E-07*

Table 2.Results for h=0.1 and λ =-1.

x	*Method Störmer*	*Method 24*	*Method Störmer 1*
0.2	*1.1E-07*	*6.2E-12*	*2.5E-09*
0.4	*6.0E-07*	*3.4E-11*	*1.4E-08*
0.6	*1.4E-07*	*8.0E-11*	*3.4E-08*
0.8	*2.4E-06*	*1.4E-10*	*6.1E-08*
1.0	*3.5E-06*	*2.0E-10*	*9.6E-08*

By using obtained results, one can confirm that if the solution of the problem decreases in the case,h $\rightarrow$ 0,then the error of the using method also decreases.

II. Conclusions

Here, we have completely compared Multistep Methods using the first derivative of the desired solution with the Multistep Second Derivative Methods using first and second derivatives of the desired solution. The obtained result of comparison has been illustrated by using specific Methods of corresponding order of accuracy. Here for receive reliable results, using Multistep has been recommended to use the bilateral methods by using the specified formulas is one of the new directions in the field of Multistep Methods. Let's note that by continue the above research one can be constructed new classes of Numerical Method with the best properties. Here, we tried to select simple methods, which can be easily applied in solving theoretical and practical problems.

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