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Fixed Point Theorem for Finite Families of κ_i-Strict Pseudo-Contractive Mappings and The Set of Solutions of Variational Inequality Problems and The Set of Solutions to Equilibrium Problems in Hilbert Space

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Abstract: - The purpose of this paper is to prove a strong convergence theorem for finding a common element of the set of solutions to equilibrium problems, the set of solutions to variational inequality problems, and the set of solutions to fixed point problems of finite families of strictly pseudo-contractive mappings. The result of this article can be especially useful in the field of engineering in terms of power systems, renewable energy, and signal processing. Moreover, I give a numerical example to support my result.

Keywords: *Fixed point problem, equilibrium problem, variational inequality problem, strictly pseudo-contractive mapping.*

I. INTRODUCTION

Throughout this article, let H be a real Hilbert space with inner product $<\cdot,\cdot>$ and norm $|\cdot|$. Let C be a nonempty closed convex subset of H. We use "$\rightharpoonup$" for weak convergence and "$\rightarrow$" for strong convergence.

Definition 1.1 Let $T: C \rightarrow C$ be a mapping. Then

(i) *the fixed point problem* for the mapping T is to find $x \in C$ such that

$$Tx = x.$$

I denote the fixed point set of a mapping T by $F(T)$.

(ii) a mapping T is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for all } x, y \in C,$$

(iii) a mapping T is called κ-*strict pseudo-contraction* if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \text{for all } x, y \in C,$$

Note that T is nonexpansive mapping if and only if T is 0-strict pseudo-contractive mapping.

Equilibrium problem theory gives us a natural and unified framework to apply a wide class of problems in various fields, such as field of CR networks, multihop communication, networks, economics, physics, finance, transportation, structural analysis, and optimization. The concepts and techniques of this theory are being used throughout the world. It has been shown by Blum and Oettli, see more detail in [1].

Let $F: C \times C \rightarrow \mathbb{R}$ be a bifunction. *The equilibrium problem* of F is to find $x \in C$, such that

$$F(x, y) \geq 0, \tag{1.1}$$

for all $y \in C$. The set of solutions of (1.1) is denoted by $EP(F)$.

The variational inequality problem is to find a point $u \in C$, such that

$$<Au, v - u> \geq 0, \tag{1.2}$$

for all $v \in C$. The set of solutions of (1.2) is denoted by $VI(C, A)$. The variational inequality offers a suitable framework for applications at least to signal processing and transportation networks. Furthermore, the variational inequality problem has been used to solve a wide range of problems in finance, regional, industry, engineering, economics, pure and applied sciences, see more detail in [2,3].

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Remark 1.2 If $T: C \to C$ be κ -strictly pseudo-contractive with $F(T) \neq \varnothing$, then $F(T) = VI(C, I - T)$, see more detail in [4].

In 2011, Kangtunyakarn [5] proved strong convergence theorem for finding a common element of the set of solution of equilibrium problem and the set of variational inequality and the set of fixed-point problems by using S-mapping generated by infinite family of nonexpansive mapping and infinite real number as follows;

Theorem 1.3 [5] Let $F: C \times C \to \mathbb{R}$ be bifunctions satisfying (A1)-(A4). Let $A: C \to H$ be a α-inverse-strongly monotone mapping. Let $\{T_i\}_{i=1}^{\infty}$ be infinite family of nonexpansive mappings with $\xi = \bigcap_{i=1}^{\infty} F(T_i) \cap EP(F) \cap VI(C, A)$, and let $\rho_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j + \alpha_2^j \leq b < 1$, and $\alpha_1^j, \alpha_2^j, \alpha_3^j \in (0, 1)$, $\forall j = 1, 2, \ldots$. For every $n \in \mathbb{N}$, let S_n and S-mappings generated by $T_n, \ldots, T_1$ and $\rho_n, \rho_{n-1}, \ldots, \rho_1$, and $T_n, T_{n-1}, \cdots$ and $\rho_n, \rho_{n-1}, \cdots$, respectively. Let $\{x_n\}$ and $\{u_n\}$ be the sequences generated by $x_1, u \in C$ and

$$\begin{cases} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \forall y \in C, \\ x_{n+1} = \alpha_n u + \beta_n P_C (I - \lambda A) x_n + \gamma_n S_n P_C (I - \lambda A) u_n, \forall n \geq 1, \end{cases} \quad (1.3)$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0, 1)$, such that $\alpha_n + \beta_n + \gamma_n = 1$, $\beta_n \in [c, d] \subset (0, 1)$, $r_n \in [a, b] \subset (0, 2\alpha)$, $\lambda \subset (0, 2\alpha)$. Assume that

(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=1}^{\infty} \alpha_1^n < \infty$,

(iii) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

Then the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z = P_\xi u$.

Many the past decades, Kangtunyakarn and Suantai [6] introduced the following a new iterative scheme for finding a common fixed point of a finite family of strict pseudo-contractive mapping: for a point $u \in H$ and $x_1 \in H$, let $\{x_n\}$ and $\{y_n\}$ be the sequences defined iteratively by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) S_n x_n, \\ x_{n+1} = \alpha_n \gamma (a_n u + (1 - a_n) f(x_n)) + (I - \alpha_n A) y_n, \forall n \geq 1, \end{cases} \quad (1.4)$$

where $\{\alpha_n\}, \{\beta_n\}, \{a_n\} \in [0, 1]$. They proved that the sequences $\{x_n\}$ and $\{y_n\}$ strongly converge to $q \in \bigcap_{i=1}^{N} F(T_i)$.

In this paper, motivated by the idea of method [6], I prove a strong convergence theorem for finding a common element of the set of solutions to equilibrium problems, the set of solutions to variational inequality problems, and the set of solutions to fixed point problems of finite families of strictly pseudo-contractive mappings. The result of this article can be especially useful in the field of engineering in terms of power systems, renewable energy, and signal processing. Moreover, I give a numerical example to support my result.

II. PRELIMINARIES

In this section, the following lemmas are important to prove our main result. Let P_C be a metric projection of H onto C i.e., for $x \in H$, $P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.1 (See [7]) Given $x \in H$ and $y \in C$. Then

$$P_C x = y \Leftrightarrow \langle x - y, y - z \rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.2 (See [8]) Let $\{s_n\}$ be a sequence of nonnegative real number satisfying

$$s_{n+1}=\left(1-\alpha_n\right)s_n+\alpha_n\beta_n, \qquad \forall n\geq 0,$$

where $\{\alpha_n\},\ \{\beta_n\}$ satisfy the conditions

(i) $\{\alpha_n\}\subset[0,1]$, $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii) $\lim_{n\to\infty} sup\ \beta_n\leq 0$ or $\sum_{n=1}^{\infty}\alpha_n\beta_n<\infty$.

(iii) Then $\lim_{n\to\infty} s_n=0$.

Lemma 2.3 (See [8]) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1}=\left(1-\alpha_n\right)s_n+\delta_n, \qquad \forall n\geq 0,$$

where $\{\alpha_n\}\subset(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii) $\lim_{n\to\infty} sup\ \frac{\delta_n}{\alpha_n}\leq 0$ or $\sum_{n=1}^{\infty}\left|\delta_n\right|<\infty$.

Then $\lim_{n\to\infty} s_n=0$.

Lemma 2.4 (See [7]) Let $A:C\to H$ be a mapping and $u\in C$, then

$$u=P_C(I-\lambda A)u\Leftrightarrow u\in VI(C,A),\ \forall\lambda>0$$

where P_C is the metric projection of H onto C.

Lemma 2.5 (See [5]) Let $A,B:C\to H$ be α and β-inverse strongly monotone mappings, respectively, with $\alpha,\beta>0$ and $VI(C,A)\cap VI(C,B)\neq\varnothing$. Then

$$VI(C,aA+(1-a)B)=VI(C,A)\cap VI(C,B), \qquad \forall a\in(0,1).$$

Furthermore, if $0<\gamma<min\{2\alpha,2\beta\}$ then $I-\gamma(aA+(1-a)B)$ is a nonexpansive mapping.

Lemma 2.6 (See [9]) Let $S:C\to C$ be a mapping. If S is a κ-strict pseudo-contractive mapping, then S satisfies the Lipschitz condition

$$\|Sx-Sy\|\leq\frac{1+\kappa}{1-\kappa}\|x-y\|, \qquad \forall x,y\in C.$$

For solving the equilibrium problems for a bifunction $F:C\times C\to\mathbb{R}$, let us assume that satisfies the following conditions:

(A1) $F(x,x)=0, \quad \forall x\in C$,

(A2) F is monotone, i.e., $F(x,y)+F(y,x)\leq 0, \quad \forall x,y\in C$,

(A3) $\forall x,y,z\in C$,

$$\lim_{t\to 0^+} F(tz+(1-t)x,y)\leq F(x,y),$$

(A4) $\forall x\in C$, $y\mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 2.7 (See [1]) Let $F:C\times C\to\mathbb{R}$ be a bifunction satisfying (A1)-(A4).

$$F(z,y)+\frac{1}{r}\left\langle y-z,z-x\right\rangle\geq 0, \quad \forall x\in C. \qquad (2.1)$$

Lemma 2.8 (See [10]) Assume that $F:C\times C\to\mathbb{R}$ satisfies (A1)-(A4). For $r>0$ and $x\in H$, define a mapping $T_r:H\to C$ as follows:

$$T_z(x)=\left\{z\in C:F(z,y)+\frac{1}{r}\left\langle y-z,z-x\right\rangle\geq 0,\ \forall y\in C\right\}, \qquad (2.2)$$

for all $z\in H$. Then, the following hold:

(i) T_r is single-valued,

(ii) T_r is firmly nonexpansive i.e.,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y \rangle, \forall x, y \in H,$$

(iii) $F(T_r) = EP(F)$,

(iv) $EP(F)$ is closed and convex.

Definition 2.9 (See [6]) Let $\{T_i\}_{i=1}^N : C \to C$ be finite family of κ_i -strict pseudo-contractions. Let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$ where $I \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\forall j = 1, 2, \ldots, N$. Define the mapping $S: C \to C$ as follows:

$$U_0 = I$$

$$U_1 = \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I$$

$$U_2 = \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I$$

$$U_3 = \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I$$

$$\vdots$$

$$U_{N-1} = \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I$$

$$S = U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.$$

This mapping is called S-mapping generated by $T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Lemma 2.10 (See [6]) Let $\{T_i\}_{i=1}^N : C \to C$ be finite family of κ_i-strict pseudo-contractive mappings with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and $\kappa = max\{\kappa_i : i = 1, 2, \ldots, N\}$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $\forall j = 1, 2, \ldots, N$, where $I = [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (\kappa, 1)$, $\forall j = 1, 2, \ldots, N-1$ and $\alpha_1^N \in (\kappa, 1]$, $\alpha_3^N, \alpha_2^j \in [\kappa, 1)$, $\forall j = 1, 2, \ldots, N$. Let S be the mapping generated b$T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a nonexpansive mapping.

Lemma 2.11 (See [11]) Let $\{T_i\}_{i=1}^N : C \to C$ be finite family of nonexpansive mappings with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j)$, $\forall j = 1, 2, \ldots, N$, where $\alpha_1^j, \alpha_2^j, \alpha_3^j \in [0, 1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j \in (0, 1)$, $\forall j = 1, 2, \ldots, N-1$, $\alpha_1^N \in (0, 1]$, $\alpha_2^j, \alpha_3^j \in [0, 1)$, $\forall j = 1, 2, \ldots, N$. Let S be the mapping generated by $T_1, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$.

III. MAIN RESULT

Theorem 3.1 Let C be a nonempty closed convex subset of a real Hilbert space H. For every $i = 1, 2, \ldots, n$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying (A1) - (A4). Let $A_i : C \to \mathbb{R}$ be α_i-inverse strongly monotones for all $i = 1, 2, \ldots, n$, with $\rho_i = (\alpha_1^i, \alpha_2^i, \alpha_3^i) \in I \times I \times I$, where $I = [0, 1]$, $\alpha_1^i + \alpha_2^i + \alpha_3^i = 1$, $\alpha_1^i + \alpha_2^i \leq e < 1$, and $\alpha_1^i, \alpha_2^i, \alpha_3^i \in (0, 1)$ for all $i = 1, 2, \ldots, N$ and let $\{P_i\}_{i=1}^N$ be κ_i-strict pseudo-contractive mappings of C into itself with $\kappa = sup_{i=1,2,\ldots,N}\{\kappa_i\}$ and let $\overline{\rho}_i = (\overline{\alpha}_1^i, \overline{\alpha}_2^i, \overline{\alpha}_3^i) \in I \times I \times I$, where $I = [0, 1]$, $\overline{\alpha}_1^i + \overline{\alpha}_2^i + \overline{\alpha}_3^i = 1$, $\overline{\alpha}_1^i + \overline{\alpha}_2^i \leq f < 1$, and $\overline{\alpha}_1^i, \overline{\alpha}_2^i, \overline{\alpha}_3^i \in (\kappa, 1)$ for all $i = 1, 2, \ldots, N$. Let S_A be S-mapping generated by $P_C(I - \delta_1 A_1), P_C(I - \delta_2 A_2), \ldots, P_C(I - \delta_N A_N)$ and $\rho_N, \rho_{N-1}, \ldots, \rho_1$ where $0 < \delta_i < 2\alpha_i$ for all $i = 1, 2, \ldots, N$. Let S_B be S-mapping generated by $P_N, P_{N-1}, \ldots, P_1$ and $\overline{\rho}_N, \overline{\rho}_{N-1}, \ldots, \overline{\rho}_1$. Assume that

$\mathscr{F}=\bigcap_{i=1}^{N} VI\left(C,A_i\right)\cap\bigcap_{i=1}^{N} F\left(P_i\right)\cap\bigcap_{i=1}^{N} EP\left(F_i\right)\neq\varnothing$ and let the sequences $\{x_n\}$ and $\{u_n^i\}$ be generated by $u,x_1\in C$ and

$$\begin{cases} F_i\left(u_n^i,v\right)+\frac{1}{r_n}\left\langle v-u_n^i,u_n^i-x_n\right\rangle\geq 0, \text{ for all } v\in C \text{ and } i=1,2,\ldots,N,\\ x_{n+1}=\alpha_n u+\beta_n P_C\left(I-\lambda\left(a\left(I-S_A\right)+(1-a)\left(I-S_B\right)\right)\right)x_n+\gamma_n\sum_{i=1}^{N}a_i u_n^i, \quad \forall n\geq 1, \end{cases} \tag{3.1}$$

for all $n\geq 1$, where $\{\alpha_n\},\ \{\beta_n\},\ \{\gamma_n\}\subseteq[0,1]$ with $\alpha_n+\beta_n+\gamma_n=1$ for all $n\in\mathbb{N}$ and $a\in(0,1)$. Suppose that the following conditions hold :

(i) $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$,

(ii) $\beta_n,\gamma_n\in[c,d]\subset(0,1)$, $\forall n\in\mathbb{N}$,

(iii) $\sum_{n=1}^{N}a_i=1$, where $a_i>0$ for all $i=1,2,\ldots,N$,

(iv) $0<a<r_n<b$, for all $n\in\mathbb{N}$,

(v) $\lambda\in(0,1-\kappa)$,

(vi) $\sum_{n=1}^{\infty}\left|\gamma_{n+1}-\gamma_n\right|<\infty$, $\sum_{n=1}^{\infty}\left|\alpha_{n+1}-\alpha_n\right|<\infty$, $\sum_{n=1}^{\infty}\left|\beta_{n+1}-\beta_n\right|<\infty$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{z}=P_{\mathscr{F}}u$.

Proof. Let $x,y\in C$. Since A_i are α_i-inverse strongly monotones and $\delta_i\leq 2\alpha_i$, I have

$$\begin{aligned} &\left\|\left(I-\delta_iA_i\right)x-\left(I-\delta_iA_i\right)y\right\|^2\\ &=\left\|x-y-\delta_i\left(A_ix-A_iy\right)\right\|^2\\ &=\left\|x-y\right\|^2-2\delta_i\left\langle x-y,A_ix-A_iy\right\rangle+\delta_i^2\left\|A_ix-A_iy\right\|^2\\ &\leq\left\|x-y\right\|^2-2\alpha_i\delta_i\left\|A_ix-A_iy\right\|^2+\delta_i^2\left\|A_ix-A_iy\right\|^2\\ &=\left\|x-y\right\|^2+\delta_i\left(\delta_i-2\alpha_i\right)\left\|A_ix-A_iy\right\|^2\\ &\leq\left\|x-y\right\|^2. \end{aligned} \tag{3.2}$$

Thus, $\left(I-\delta_iA_i\right)$ are nonexpansive mappings, for all $i=1,2,\ldots,N$. Hence, $P_C\left(I-\delta_iA_i\right)$ are nonexpansive mappings, for all $i=1,2,\ldots,N$.

I will divide my proof into 5 steps.

Step 1. I will show that the sequence $\{x_n\}$ is bounded. Since

$$F_i\left(u_n^i,v\right)+\frac{1}{r_n}\left\langle v-u_n^i,u_n^i-x_n\right\rangle\geq 0, \text{ for all } v\in C \text{ and } i=1,2,\ldots,N.$$

By Lemma 2.8, I have $u_n^i=T_{r_n}^i\left(x_n\right)$ and $EP\left(F_i\right)=F\left(T_{r_n}^i\right)$, for all $i=1,2,\ldots,N$.

Let $z\in\mathscr{F}=\bigcap_{i=1}^{N} VI\left(C,A_i\right)\cap\bigcap_{i=1}^{N} F\left(P_i\right)\cap\bigcap_{i=1}^{N} EP\left(F_i\right)$.

Since $P_C\left(I-\delta_iA_i\right)$ are nonexpansive mappings, for all $i=1,2,\ldots,N$. By Lemma 2.4, 2.5, 2.10, 2.11, I get $z\in VI\left(C,a\left(I-S_A\right)+(1-a)\left(I-S_B\right)\right)$.

From Lemma 2.4, I have

$$z=P_C\left(I-\lambda\left(a\left(I-S_A\right)+(1-a)\left(I-S_B\right)\right)\right)z. \tag{3.3}$$

From nonexpansiveness of $T_{r_n}^i$ and (3.3), I have

$$
\begin{aligned}
&\left\|x_{n+1}-z\right\| \\
&=\left\|\alpha_n u+\beta_n P_C\left(I-\lambda\left(a\left(I-S_A\right)+(1-a)\left(I-S_B\right)\right)\right)x_n+\gamma_n\sum_{i=1}^{N}a_i u_n^i-z\right\| \\
&\leq \alpha_n\|u-z\|+\beta_n\left\|P_C\left(I-\lambda\left(a\left(I-S_A\right)+(1-a)\left(I-S_B\right)\right)\right)x_n-z\right\| \\
&\quad+\gamma_n\left\|\sum_{i=1}^{N}a_i\left(u_n^i-z\right)\right\| \\
&\leq \alpha_n\|u-z\|+\beta_n\left\|x_n-z\right\|+\gamma_n\left\|x_n-z\right\| \\
&\leq \alpha_n\|u-z\|+\left(1-\alpha_n\right)\left\|x_n-z\right\|.
\end{aligned} \tag{3.4}
$$

Putting $J=max\left\{\|u-z\|,\left\|x_1-z\right\|\right\}$. By induction and (3.4), I have $\left\|x_n-z\right\|\leq J$, for all $n\in\mathbb{N}$. It implies that $\{x_n\}$ and $\{u_n^i\}$ are bounded, for all $i=1,2,\ldots,N$.

Step 2. I will show that $\lim_{n\to\infty}\left\|x_{n+1}-x_n\right\|=0.$

Putting $M=a\left(I-S_A\right)+(1-a)\left(I-S_B\right)$. From the definition of $\{x_n\}$, I get

$$
\begin{aligned}
&\left\|x_{n+1}-x_n\right\| \\
&=\Big\|\alpha_n u+\beta_n P_C\left(I-\lambda\left(a\left(I-S_A\right)+(1-a)\left(I-S_B\right)\right)\right)x_n+\gamma_n\sum_{i=1}^{N}a_i u_n^i \\
&\quad-\alpha_{n-1}u+\beta_{n-1}P_C\left(I-\lambda\left(a\left(I-S_A\right)+(1-a)\left(I-S_B\right)\right)\right)x_{n-1}+\gamma_{n-1}\sum_{i=1}^{N}a_i u_{n-1}^i\Big\| \\
&\leq\left|\alpha_n-\alpha_{n-1}\right|\|u\|+\beta_n\left\|x_n-x_{n-1}\right\|+\left|\beta_n-\beta_{n-1}\right|\left\|P_C(I-\lambda M)x_{n-1}\right\|+\left|\gamma_n-\gamma_{n-1}\right|\left\|\sum_{i=1}^{N}a_i u_{n-1}^i\right\|+\gamma_n\sum_{i=1}^{N}a_i\left\|u_n^i-u_{n-1}^i\right\|.
\end{aligned} \tag{3.5}
$$

Since $u_n^i=T_{r_n}^i\left(x_n\right)$ and the definition of $T_{r_n}^i\left(x_n\right)$, I have

$$
F_i\left(T_{r_n}^i x_n,v\right)+\frac{1}{r_n}\left\langle v-T_{r_n}^i x_n,T_{r_n}^i x_n-x_n\right\rangle\geq 0,\quad\forall v\in C. \tag{3.6}
$$

Similarly,

$$
F_i\left(T_{r_{n+1}}^i x_{n+1},v\right)+\frac{1}{r_{n+1}}\left\langle v-T_{r_{n+1}}^i x_{n+1},T_{r_{n+1}}^i x_{n+1}-x_{n+1}\right\rangle\geq 0,\forall v\in C. \tag{3.7}
$$

From (3.6) and (3.7), I obtain

$$
F_i\left(T_{r_n}^i x_n,T_{r_{n+1}}^i x_{n+1}\right)+\frac{1}{r_n}\left\langle T_{r_{n+1}}^i x_{n+1}-T_{r_n}^i x_n,T_{r_n}^i x_n-x_n\right\rangle\geq 0, \tag{3.8}
$$

and

$$
F_i\left(T_{r_{n+1}}^i x_{n+1},T_{r_n}^i x_n\right)+\frac{1}{r_{n+1}}\left\langle T_{r_n}^i x_n-T_{r_{n+1}}^i x_{n+1},T_{r_{n+1}}^i x_{n+1}-x_{n+1}\right\rangle\geq 0. \tag{3.9}
$$

By (3.8) and (3.9), I get

$$
\frac{1}{r_n}\left\langle T_{r_{n+1}}^i x_{n+1}-T_{r_n}^i x_n,T_{r_n}^i x_n-x_n\right\rangle+\frac{1}{r_{n+1}}\left\langle T_{r_n}^i x_n-T_{r_{n+1}}^i x_{n+1},T_{r_{n+1}}^i x_{n+1}-x_{n+1}\right\rangle\geq 0.
$$

It follows that

$$
\left\langle T_{r_n}^i x_n-T_{r_{n+1}}^i x_{n+1},\frac{T_{r_{n+1}}^i x_{n+1}-x_{n+1}}{r_{n+1}}-\frac{T_{r_n}^i x_n-x_n}{r_n}\right\rangle\geq 0.
$$

This implies that

$$
\left\langle T_{r_{n+1}}^i x_{n+1}-T_{r_n}^i x_n,T_{r_n}^i x_n-T_{r_{n+1}}^i x_{n+1}+T_{r_{n+1}}^i x_{n+1}-x_n-\frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}^i x_{n+1}-x_{n+1}\right)\right\rangle\geq 0. \tag{3.10}
$$

It follows that

$$
\begin{aligned}
&\left\|T_{r_{n+1}}^i x_{n+1}-T_{r_n}^i x_n\right\|^2 \\
&\quad\leq\left\langle T_{r_{n+1}}^i x_{n+1}-T_{r_n}^i x_n,T_{r_{n+1}}^i x_{n+1}-x_n-\frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}^i x_{n+1}-x_{n+1}\right)\right\rangle \\
&\quad\leq\left\|T_{r_{n+1}}^i x_{n+1}-T_{r_n}^i x_n\right\|\left\|x_{n+1}-x_n+\left(1-\frac{r_n}{r_{n+1}}\right)\left(T_{r_{n+1}}^i x_{n+1}-x_{n+1}\right)\right\| \\
&\quad\leq\left\|T_{r_{n+1}}^i x_{n+1}-T_{r_n}^i x_n\right\|\left(\left\|x_{n+1}-x_n\right\|+\frac{1}{r_{n+1}}\left|r_{n+1}-r_n\right|\left\|T_{r_{n+1}}^i x_{n+1}-x_{n+1}\right\|\right) \\
&\quad\leq\left\|T_{r_{n+1}}^i x_{n+1}-T_{r_n}^i x_n\right\|\left(\left\|x_{n+1}-x_n\right\|+\frac{1}{a}\left|r_{n+1}-r_n\right|\left\|T_{r_{n+1}}^i x_{n+1}-x_{n+1}\right\|\right).
\end{aligned} \tag{3.11}
$$

Then,

$$\left\|u_{n+1}^{i}-u_{n}^{i}\right\| \leq \left\|x_{n+1}-x_{n}\right\|+\frac{1}{a}\left|r_{n+1}-r_{n}\right|\left\|u_{n+1}^{i}-x_{n+1}\right\|. \quad (3.12)$$

Substituting (3.12) into (3.5), I have

$$\begin{aligned}
&\left\|x_{n+1}-x_{n}\right\| \\
&\leq \left|\alpha_{n}-\alpha_{n-1}\right|\|u\|+\beta_{n}\left\|x_{n}-x_{n-1}\right\|+\left|\beta_{n}-\beta_{n-1}\right|\left\|P_{C}(I-\lambda M) x_{n-1}\right\| \\
&\quad+\left|\gamma_{n}-\gamma_{n-1}\right|\left\|\sum_{i=1}^{N} a_{i} u_{n-1}^{i}\right\|+\gamma_{n} \sum_{i=1}^{N} a_{i}\left\|u_{n}^{i}-u_{n-1}^{i}\right\| \\
&\leq \left|\alpha_{n}-\alpha_{n-1}\right|\|u\|+\beta_{n}\left\|x_{n}-x_{n-1}\right\|+\left|\beta_{n}-\beta_{n-1}\right|\left\|P_{C}(I-\lambda M) x_{n-1}\right\| \\
&\quad+\left|\gamma_{n}-\gamma_{n-1}\right|\left\|\sum_{i=1}^{N} a_{i} u_{n-1}^{i}\right\|+\gamma_{n} \sum_{i=1}^{N} a_{i}\left(\left\|x_{n}-x_{n-1}\right\|+\frac{1}{a}\left|r_{n}-r_{n-1}\right|\left\|u_{n}^{i}-x_{n}\right\|\right) \\
&=\left|\alpha_{n}-\alpha_{n-1}\right|\|u\|+\left(1-\alpha_{n}\right)\left\|x_{n}-x_{n-1}\right\|+\left|\beta_{n}-\beta_{n-1}\right|\left\|P_{C}(I-\lambda M) x_{n-1}\right\| \\
&\quad+\left|\gamma_{n}-\gamma_{n-1}\right|\left\|\sum_{i=1}^{N} a_{i} u_{n-1}^{i}\right\|+\frac{\gamma_{n}}{a}\left|r_{n}-r_{n-1}\right| \sum_{i=1}^{N} a_{i}\left\|u_{n}^{i}-x_{n}\right\| \\
&\leq \left|\alpha_{n}-\alpha_{n-1}\right| \overline{M_{1}}+\left(1-\alpha_{n}\right)\left\|x_{n}-x_{n-1}\right\|+\left|\beta_{n}-\beta_{n-1}\right| \overline{M_{1}} \\
&\quad+\left|\gamma_{n}-\gamma_{n-1}\right| \overline{M_{1}}+\frac{\gamma_{n}}{a}\left|r_{n}-r_{n-1}\right| \overline{M_{1}}, \quad (3.13)
\end{aligned}$$

where $\overline{M_{1}}=\max_{n \in \mathbb{N}}\left\{\|u\|,\left\|x_{n}\right\|,\left\|\sum_{i=1}^{N} a_{i} u_{n}^{i}\right\|, \sum_{i=1}^{N} a_{i}\left\|u_{n}^{i}-x_{n}\right\|\right\}$.

From Lemma 2.3, (3.13), and the conditions (i)-(vi), I obtain

$$\lim_{n \rightarrow \infty}\left\|x_{n+1}-x_{n}\right\|=0. \quad (3.14)$$

Step 3. I will show that $\lim_{n \rightarrow \infty}\left\|u_{n}^{i}-x_{n}\right\|=0$ and $\lim_{n \rightarrow \infty}\left\|P_{C}(I-\lambda M) x_{n}-x_{n}\right\|=0$, where $M=a\left(I-S_{A}\right)+(1-a)\left(I-S_{B}\right)$, for all $a \in(0,1)$.

Since $u_{n}^{i}=T_{r_{n}}^{i}\left(x_{n}\right)$ and $T_{r_n}^{i}$ is firmly nonexpansive mappings, I have

$$\left\|z-T_{r_{n}}^{i} x_{n}\right\|^{2}=\left\|T_{r_{n}}^{i} x_{n}-T_{r_{n}}^{i} x_{n}\right\|^{2} \leq \frac{1}{2}\left(\left\|T_{r_{n}}^{i} x_{n}-z\right\|^{2}+\left\|x_{n}-z\right\|^{2}-\left\|T_{r_{n}}^{i} x_{n}-z\right\|^{2}\right),$$

then

$$\left\|u_{n}^{i}-z\right\|^{2} \leq\left\|x_{n}-z\right\|^{2}-\left\|u_{n}^{i}-x_{n}\right\|^{2}. \quad (3.15)$$

From Lemma 2.5, (3.15), and the definition of $\{x_n\}$, I have

$$\begin{aligned}
\left\|x_{n+1}-z\right\|^{2} &\leq \alpha_{n}\|u-z\|^{2}+\beta_{n}\left\|x_{n}-z\right\|^{2}+\gamma_{n} \sum_{i=1}^{N} a_{i}\left\|u_{n}^{i}-z\right\|^{2} \\
&\leq \alpha_{n}\|u-z\|^{2}+\beta_{n}\left\|x_{n}-z\right\|^{2}+\gamma_{n} \sum_{i=1}^{N} a_{i}\left(\left\|x_{n}-z\right\|^{2}-\left\|u_{n}^{i}-x_{n}\right\|^{2}\right) \\
&\leq \alpha_{n}\|u-z\|^{2}+\left\|x_{n}-z\right\|^{2}-\gamma_{n} \sum_{i=1}^{N} a_{i}\left\|u_{n}^{i}-x_{n}\right\|^{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\gamma_{n} \sum_{i=1}^{N} a_{i}\left\|u_{n}^{i}-x_{n}\right\|^{2} &\leq \alpha_{n}\|u-z\|^{2}+\left\|x_{n}-z\right\|^{2}-\left\|x_{n+1}-z\right\|^{2} \\
&\leq \alpha_{n}\|u-z\|^{2}+\left\|x_{n}-x_{n+1}\right\|\left(\left\|x_{n}-z\right\|+\left\|x_{n+1}-z\right\|\right).
\end{aligned}$$

By (3.14), (3.16), and the condition (i), I have

$$\lim_{n \rightarrow \infty}\left\|u_{n}^{i}-x_{n}\right\|=0, \quad \text{for all } i=1,2, \ldots, N. \quad (3.17)$$

From the nonexpansiveness of P_C, I obtain

$$\begin{aligned}
&\left\|P_{C}(I-\lambda M) x_{n}-z\right\|^{2} \\
&\leq\left\|x_{n}-z-\lambda\left(M x_{n}-M z\right)\right\|^{2} \\
&\leq\left\|x_{n}-z\right\|^{2}-2 \lambda\left\langle x_{n}-z, M x_{n}-M z\right\rangle+\left\|\lambda\left(M x_{n}-M z\right)\right\|^{2}. \quad (3.18)
\end{aligned}$$

From (3.18), I have

$$\begin{aligned}
&\left\|P_{C}(I-\lambda M) x_{n}-z\right\|^{2} \\
&\leq\left\|x_{n}-z\right\|^{2}-\lambda\left\|M x_{n}-M z\right\|^{2}+\lambda^{2}\left\|M x_{n}-M z\right\|^{2} \\
&=\left\|x_{n}-z\right\|^{2}-\lambda(1-\lambda)\left\|M x_{n}-M z\right\|^{2}, \quad (3.19)
\end{aligned}$$

By the definition of x_n and (3.19), I get

$$\begin{aligned}&\|x_{n+1}-z\|^2\\&\leq \alpha_n\|u-z\|^2+\beta_n\left[\|x_n-z\|^2-\lambda(1-\lambda)\|Mx_n-Mz\|^2\right]+\gamma_n\|x_n-z\|^2\\&\leq \alpha_n\|u-z\|^2+\|x_n-z\|^2-\lambda\beta_n(1-\lambda)\|Mx_n-Mz\|^2.\end{aligned}$$

Then

$$\begin{aligned}&\lambda\beta_n(1-\lambda)\|Mx_n-Mz\|^2\\&\leq \alpha_n\|u-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2\\&\leq \alpha_n\|u-z\|^2+\|x_n-x_{n+1}\|\left(\|x_n-z\|+\|x_{n+1}-z\|\right).\end{aligned} \tag{3.20}$$

From (3.14), (3.20), and the condition (i), I have

$$\lim_{n\to\infty}\|Mx_n-Mz\|=0. \tag{3.21}$$

From the definition of $P_C(I-\lambda M)$ and Lemma 2.5, I have

$$\begin{aligned}&\|P_C(I-\lambda M)x_n-z\|^2\\&\leq \|P_C(I-\lambda M)x_n-P_C(I-\lambda M)z\|^2\\&\leq \frac{1}{2}\left(\|x_n-z\|^2+\|P_C(I-\lambda M)x_n-z\|^2-\|x_n-P_C(I-\lambda M)x_n-\lambda\left(Mx_n-Mz\right)\|^2\right)\\&\leq \frac{1}{2}\left(\|x_n-z\|^2+\|P_C(I-\lambda M)x_n-z\|^2-\|x_n-P_C(I-\lambda M)x_n\|^2\right)\\&\quad+\frac{1}{2}\left(2\lambda\|x_n-P_C(I-\lambda M)x_n\|\|Mx_n-Mz\|\right).\end{aligned}$$

It follows that

$$\|P_C(I-\lambda M)x_n-z\|^2\leq \|x_n-z\|^2-\|x_n-P_C(I-\lambda M)x_n\|^2+2\lambda\|x_n-P_C(I-\lambda M)x_n\|\|Mx_n-Mz\|. \tag{3.22}$$

By the definition of x_n, (3.22), and the nonexpansiveness of $T^i_{r_n}$, I get

$$\begin{aligned}&\|x_{n+1}-z\|^2\\&\leq \alpha_n\|u-z\|^2+\beta_n\|P_C(I-\lambda M)x_n-z\|^2+\gamma_n\|x_n-z\|^2\\&\leq \alpha_n\|u-z\|^2+\beta_n\left[\|x_n-z\|^2-\|x_n-P_C(I-\lambda M)x_n\|^2\right]\\&\quad+2\beta_n\lambda\|x_n-P_C(I-\lambda M)x_n\|\|Mx_n-Mz\|+\gamma_n\|x_n-z\|^2\\&\leq \alpha_n\|u-z\|^2+\|x_n-z\|^2-\beta_n\|x_n-P_C(I-\lambda M)x_n\|^2\\&\quad+2\lambda\|x_n-P_C(I-\lambda M)x_n\|\|Mx_n-Mz\|,\end{aligned}$$

then

$$\begin{aligned}&\beta_n\|x_n-P_C(I-\lambda M)x_n\|^2\\&\leq \alpha_n\|u-z\|^2+\|x_n-z\|^2-\|x_{n+1}-z\|^2+2\lambda\|x_n-P_C(I-\lambda M)x_n\|\|Mx_n-Mz\|\\&\leq \alpha_n\|u-z\|^2+\left(\|x_n-z\|-\|x_{n+1}-z\|\right)\|x_{n+1}-x_n\|\\&\quad+2\lambda\|x_n-P_C(I-\lambda M)x_n\|\|Mx_n-Mz\|.\end{aligned} \tag{3.23}$$

From condition (i), (3.14), (3.21), and (3.23), I obtain

$$\lim_{n\to\infty}\|x_n-P_C(I-\lambda M)x_n\|=0. \tag{3.24}$$

Step 4. I will show that $\lim_{n\to\infty} sup\left\langle u-\bar{z},x_n-\bar{z}\right\rangle\leq 0$, where $\bar{z}=P_{\mathscr{F}}u$.

Take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, such that

$$\lim_{n\to\infty} sup\left\langle u-\bar{z},x_n-\bar{z}\right\rangle=\lim_{n\to\infty}\left\langle u-\bar{z},x_{n_k}-\bar{z}\right\rangle \tag{3.25}$$

Without loss of generality, I may assume that $x_{n_k}\rightharpoonup \upsilon$ as $k\to\infty$, where $\upsilon\in C$.

From (3.17), I have $u^i_{n_k}\rightharpoonup\upsilon$ as $k\to\infty$, for all $i=1,2,\ldots,N$. Assume that $\upsilon\neq P_C(I-\lambda M)\upsilon$, where $M=a\left(I-S_A\right)+(1-a)\left(I-S_B\right)$.

By the nonexpansiveness of $P_C(I-\lambda M)$, (3.24), and Opial's property, I have

$$\lim_{k\to\infty} inf \left\|x_{n_k} - \upsilon\right\| < \lim_{k\to\infty} inf \left\|x_{n_k} - P_C(I - \lambda M)\upsilon\right\| \le \lim_{k\to\infty} inf \left(\left\|x_{n_k} - \upsilon\right\| \right).$$

This is a contradiction, then I have

$$\upsilon \in F\left(P_C(I - \lambda M) \right). \tag{3.26}$$

From Lemma 2.4, Lemma 2.5, and Lemma 2.11, I have

$$F\left(P_C\left(I - \lambda\left(a\left(I - S_A\right) + (1 - a)\left(I - S_B\right)\right)\right)\right) = VI\left(C, \left(I - S_A\right)\right) \cap VI\left(C, \left(I - S_B\right)\right) = \bigcap_{i=1}^{N} VI\left(C, A_i\right) \cap \bigcap_{i=1}^{N} F\left(P_i\right). \tag{3.27}$$

From (3.26) and (3.27), I have

$$\upsilon \in \bigcap_{i=1}^{N} VI\left(C, A_i\right) \cap \bigcap_{i=1}^{N} F\left(P_i\right).$$

Since

$$F_i\left(u_n^i, w\right) + \frac{1}{r_n}\left\langle w - u_n^i, u_n^i - x_n\right\rangle \ge 0,$$

for all $w \in C$ and $i = 1, 2, \ldots, N$. By (A2), I get

$$\frac{1}{r_n}\left\langle w - u_n^i, u_n^i - x_n\right\rangle \ge F_i\left(u_n^i, w\right), \quad \forall w \in C.$$

In particular

$$\left\langle w - u_{n_k}^i, \frac{1}{r_{n_k}}\left(u_{n_k}^i - x_{n_k}\right) \right\rangle \ge F_i\left(u_{n_k}^i, w\right),$$

for all $w \in C$ and $i = 1, 2, \ldots, N$.

From (A4) and (3.17), I have

$$F_i(w, \upsilon) \le 0, \tag{3.28}$$

for all $w \in C$ and $i = 1, 2, \ldots, N$.

Let $u_t := tw + (1 - t)\upsilon$, for all $t \in (0, 1]$, I have $u_t \in C$ and from (A1), (A4), and (3.28), I get

$$0 = F_i\left(u_t, u_t\right) \le tF_i\left(u_t, w\right) + (1 - t) F_i\left(u_t, \upsilon\right) \le tF_i\left(u_t, w\right),$$

for all $i = 1, 2, \ldots, N$.

Hence, $F_i(tw + (1 - t)\upsilon, w) \ge 0$, for all $t \in (0, 1]$, $w \in C$.

Letting $t \to 0^+$ and using assumption (A3), I can conclude that

$$F_i(\upsilon, w) \le 0,$$

for all $w \in C$ and $i = 1, 2, \ldots, N$.

Therefore, $\upsilon \in \bigcap_{i=1}^{N} EP\left(F_i\right)$. Hence, $\upsilon \in \mathscr{F}$.

Since $x_{n_k} \rightharpoonup \upsilon$ and $\upsilon \in \mathscr{F}$, I have

$$\lim_{n\to\infty} sup\left\langle u - \bar{z}, x_n - \bar{z}\right\rangle = \lim_{k\to\infty}\left\langle u - \bar{z}, x_{n_k} - \bar{z}\right\rangle = \left\langle u - \bar{z}, \upsilon - \bar{z}\right\rangle \le 0. \tag{3.29}$$

Step 5. Finally, I show that $\lim_{n\to\infty} x_n = \bar{z}$, where $\bar{z} = P_{\mathscr{F}} u$.

By nonexpansive of $P_C(I - \lambda M)$, I have

$$\left\|x_{n+1} - \bar{z}\right\|^2 = \left\|\alpha_n(u - \bar{z}) + \beta_n\left(P_C(I - \lambda M) x_n - \bar{z}\right) + \gamma_n\left(\sum_{i=1}^{N} a_i u_n^i - \bar{z}\right)\right\|^2 \tag{3.30}$$

$$\leq \beta_n \left\| \left(P_C(I-\lambda M)x_n - \bar{z} \right) \right\|^2 + \gamma_n \left\| \sum_{i=1}^{N} a_i u_n^i - \bar{z} \right\|^2 + 2\alpha_n \left\langle u - \bar{z}, x_{n+1} - \bar{z} \right\rangle$$
$$\leq \left(1-\alpha_n\right) \left\| x_n - \bar{z} \right\|^2 + 2\alpha_n \left\langle u - \bar{z}, x_{n+1} - \bar{z} \right\rangle.$$

From (3.29) and Lemma 2.2, I have $\{x_n\}$ converges strongly to $\bar{z} = P_{\mathscr{F}} u$.
This completes the proof of Theorem 3.1. □

IV. Example And Numerical Result

Example 4.1 Let $\mathbb{R}$ be the set of real numbers, and let $\langle \cdot , \cdot \rangle : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be an inner product defined by $\langle x, y \rangle = x \cdot y$, for all $x, y \in \mathbb{R}$ and let a usual norm $\| \cdot \| : \mathbb{R} \to \mathbb{R}$ be defined by $\|x\| = |x|$, for all $x \in \mathbb{R}$. Let $H = \mathbb{R}$ and $C = [-50, 50]$. For every $i = 1, 2, \ldots, N$, let $F_i : [-50, 50] \times [-50, 50] \to [-50, 50]$ be defined by

$F_i(x, y) = i\left(-5x^2 + 4xy + y^2\right)$, for all $x, y \in \mathbb{R}$.

Let $A_i : [-50, 50] \to [-50, 50]$ be defined by

$A_i(x) = \dfrac{2x}{3i}$, for all $x \in \mathbb{R}$.

Let $P_i : [-50, 50] \to [-50, 50]$ be defined by

$P_i(x) = \dfrac{x}{2i}$, for all $x \in \mathbb{R}$.

Then, $\bigcap_{i=1}^{N} VI\left(C, A_i\right) \cap \bigcap_{i=1}^{N} F\left(P_i\right) \cap \bigcap_{i=1}^{N} EP\left(F_i\right) = \{0\}$.

Let $u \in C$ and $\{x_n\}, \{u_n^i\}$ be the sequences generated by (3.1), for all $i = 1, 2, \ldots, N$. By the definition of F_i, I have

$$0 \leq F_i\left(u_n^i, v\right) + \frac{1}{r_n} \left\langle v - u_n^i, u_n^i - x_n \right\rangle,$$

for all $i = 1, 2, \ldots, N$ and $n \in \mathbb{N}$.

Choose $r_n = 1$, I get

$$0 \leq F_i\left(u_n^i, v\right) + \frac{1}{r_n} \left\langle v - u_n^i, u_n^i - x_n \right\rangle$$
$$= i\left[-5\left(u_n^i\right)^2 + 4u_n^i v + v^2\right] + \left(v - u_n^i\right)\left(u_n^i - x_n\right)$$
$$= -5i\left(u_n^i\right)^2 + 4iu_n^i v + iv^2 + u_n^i v - x_n v - \left(u_n^i\right)^2 + x_n u_n^i$$
$$= iv^2 + \left((1+4i)u_n^i - x_n\right)v + (-5i-1)\left(u_n^i\right)^2 + x_n u_n^i.$$

Let $Q(v) = iv^2 + \left((1+4i)u_n^i - x_n\right)v + (-5i-1)\left(u_n^i\right)^2 + x_n u_n^i$. $Q(v)$ is a quadratic function of v with coefficient $a = i$, $b = (1+4i)u_n^i - x_n$, $c = (-5i-1)\left(u_n^i\right)^2 + x_n u_n^i$. Determine the discriminant Δ of Q as follows:

$$\Delta = b^2 - 4ac$$
$$= \left[(1+4i)u_n^i - x_n\right]^2 - 4(i)\left((-5i-1)\left(u_n^i\right)^2 + x_n u_n^i\right)$$
$$= \left((6i+1)u_n^i - x_n\right)^2.$$

I know that $Q(v) \geq 0$, $\forall v \in \mathbb{R}$. If it has at most one solution in $\mathbb{R}$, then $\Delta \leq 0$, so I obtain that

$$u_n^i = \frac{x_n}{6i+1}, \tag{4.1}$$

for all $i = 1, 2, \ldots, N$.

Put $\alpha_n = \dfrac{1}{5n}$, $\beta_n = \dfrac{3n-1}{10n}$, $\gamma_n = \dfrac{7n-1}{10n}$, $\lambda = \dfrac{1}{3}$, $a = \dfrac{1}{2}$, $\delta_i = 2$, $\rho_i = \left(\alpha_1^i, \alpha_2^i, \alpha_3^i\right) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and $\bar{\rho}_i = \left(\bar{\alpha}_1^i, \bar{\alpha}_2^i, \bar{\alpha}_3^i\right) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$, for all $i = 1, 2, \ldots, N$.

From (4.1), I rewrite (3.1) as follows:

$$x_{n+1}=\left(\frac{1}{5n}\right)u+\left(\frac{3n-1}{10n}\right)P_{[-50,50]}\left(I-\frac{1}{3}\left(\frac{1}{2}\left(I-S_A\right)+\frac{1}{2}\left(I-S_B\right)\right)\right)x_n$$
$$+\left(\frac{7n-1}{10n}\right)\sum_{n=1}^{N}\left(\frac{1}{3^i}+\frac{1}{2N}+\frac{1}{2N3^N}\right)\frac{x_n}{6i+1}. \quad (4.2)$$

It is clear that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy all the conditions of Theorem 3.1, I can conclude that the sequences $\{x_n\}$ and $\{u_n^i\}$ converge strongly to 0. Table 1 and Figure 1 show the values of the sequences $\{u_n^i\}$ and $\{x_n\}$, where $u=1$ and $x_1=10$.

Table 1 The values of $\{u_n^i\}$ and $\{x_n\}$ with $u=1$, $x_1=10$, and $n=N=50$

n	u_n^i	x_n
1	0.033223	10
2	0.007992	2.405524
3	0.002437	0.733438
⋮	⋮	⋮
25	0.000042	0.012541
⋮	⋮	⋮
48	0.000021	0.006374
49	0.000021	0.006241
50	0.000020	0.006113

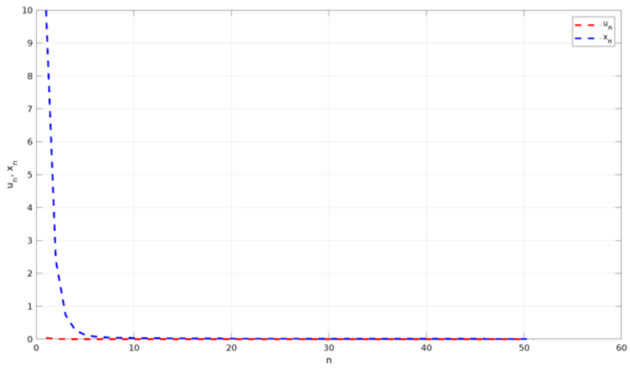

Figure 1 The convergence of $\{u_n^i\}$ and $\{x_n\}$ with $u=1$, $x_1=10$, and $n=N=50$

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