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The Tail Characteristics of Several Asymmetric Distributions and Their Applications in the Field of Financial Data Modelling and Machine Learning

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Abstract: - Asymmetric distributions play a pivotal role in various fields, including finance, machine learning, and artificial intelligence. In finance, asymmetric distributions, characterized by negative skewness, peakedness, heavy tails, and asymmetric fluctuations, provide a more accurate representation of financial data. These distributions are particularly useful in capturing the complex behavior of assets and risk management. In the field of machine learning, data disturbance, noise, outliers and other disturbances will have a negative impact on the performance and reliability of the model, asymmetric distributions offer a valuable tool for describing the tail behavior of noise and interference. This is crucial in designing more robust machine learning models that can handle outliers and noise effectively. Additionally, when dealing with unbalanced data classification, asymmetric distributions can be leveraged by adjusting the decision threshold of classifiers or employing specialized loss functions. In this paper, the asymptotic tail ratio behavior of probability density function(pdf) and cumulative distribution function (cdf) of three asymmetric distributions (asymmetric Laplace, generalized Logistic and asymmetric normal distribution) and Student-t distribution were considered respectively under some regular conditions. The detailed characteristics of these asymmetric tails will have a profound impact on the fields of financial data analysis, artificial intelligence, and unbalanced classification.

Keywords: Asymmetric Laplace Distribution, Generalized Logistic Distribution, Asymmetric Normal Distribution, Student-T Distribution, Financial Data, Machine Learning.

I. INTRODUCTION

Modeling high frequency financial time series such as interest rate and currency exchange rate, one always encounters the general characteristics of the distributions of considered time series such as asymmetry, sharp peaks and heavier tails. In order to characterize the nature of considered time series, heavy-tailed distributions and some asymmetric distributions are proposed. Some scholars have studied the tail index estimation of heavy-tailed distribution in theory and practice [1-3]. The common asymmetric distribution is also the focus of the study, such as asymmetric Laplace distribution [4-5], generalized Logistic distribution [6-7] and asymmetric normal distribution [8-9].

In the field of machine learning, we call the sensitivity of a model to noise and disturbance model robustness. The asymmetric distribution can be used to describe the tail behavior of noise and interference, thus helping to design more robust machine learning models. For example, in the field of natural language processing, asymmetric distributions can be used to design more robust text classification models.

Previous studies have mostly focused on the typical features such as sharp peak and thick tail. In this paper, we will consider the asymptotic tail ratio of probability density function (*pdf*) and cumulative distribution function (*cdf*) of asymmetric distributions and Student's *t*-distribution as the degree of freedom of the latter tending to infinity. For symmetric distribution, this kind of work has been done by Finner et al., they got some interesting results such as the asymptotic Mills' ratio of *t*-distribution and a large deviation theorem were obtained [10].

Firstly we introduce the *pdf* and *cdf* of three asymmetric distributions mentioned above. The characteristic function of asymmetric Laplace distributions (Notation AL) is given by

$$\varphi(t)=\frac{1}{1+\sigma^2t^2-i\mu t},\sigma>0,-\infty<\mu<\infty \tag{1}$$

which can be formed by the characteristic function of difference of two in- dependent exponential variables. Hence the *pdf* and *cdf* of AL distribution are [11]:

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$$al_{\sigma,\mu}(x) = \frac{\kappa}{\sigma(1+\kappa^2)} \begin{cases} \exp(-\frac{\kappa x}{\sigma}), \; x \ge 0 \\ \exp(\frac{x}{\sigma\kappa}), \; x < 0 \end{cases} \tag{2}$$

and

$$AL_{\sigma,\mu}(x) = \begin{cases} 1 - \frac{1}{1+\kappa^2}\exp(-\frac{\kappa x}{\sigma}), \; x \ge 0 \\ \frac{\kappa^2}{1+\kappa^2}\exp(\frac{x}{\sigma\kappa}), \; x < 0 \end{cases} \tag{3}$$

where $\kappa = (2\sigma)/(\mu + \sqrt{4\sigma^2 + \mu^2})$ and $\kappa / \sigma > 0$.

Difference of two independent Gumbel-distributed random variables has the standard logistic distribution [6-7] .The *pdf* and *cdf* of generalized Logistic distribution (Notation GL) are:

$$gl_{b,\sigma}(x) = \frac{b\exp(-\frac{x}{\sigma})}{\sigma(1+\exp(-\frac{x}{\sigma}))^{b+1}} \tag{4}$$

and

$$GL_{b,\sigma}(x) = (1+\exp(-\frac{x}{\sigma}))^{-b} \tag{5}$$

where $b/\sigma > 0$.

Earlier work on the asymmetric normal distribution were presented by Azzalini and Dalla [8]. Bennett [9] also studied some properties of asymmetric normal distribution. The *pdf* of asymmetric normal distribution (Notation AN) is given by:

$$an_{\theta,\sigma_l,\sigma_r}(x) = \frac{2}{\sqrt{2\pi}(\sigma_l+\sigma_r)} \begin{cases} \exp\{-\frac{(x-\theta)^2}{2\sigma_l^2}\}, x \le \theta \\ \exp\{-\frac{(x-\theta)^2}{2\sigma_r^2}\}, x > \theta \end{cases} \tag{6}$$

where $\sigma_l, \sigma_r > 0, \theta \in R$.

An illustration of *pdf* of AL, GL and AN with different parameters, see Figure 1-3.

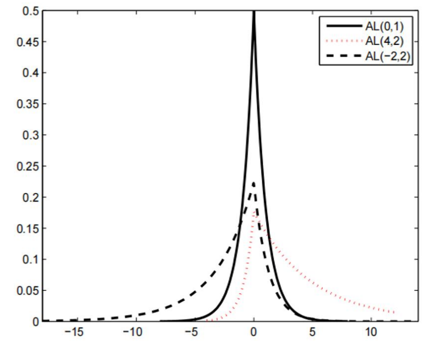

Figure 1: Asymmetric Laplace Probability Density

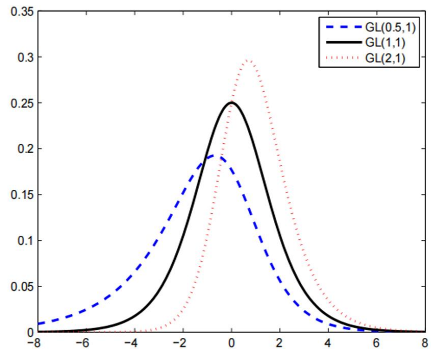

Figure 2: Generalized Logistic Probability Density

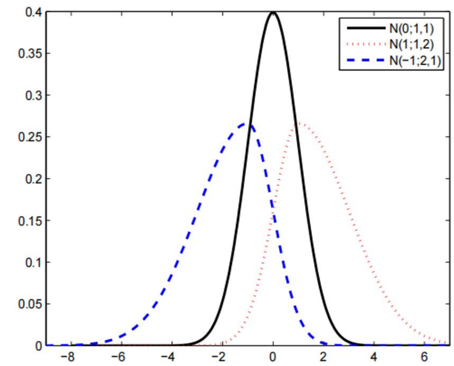

Figure 3: Asymmetric Normal Distribution Density

Now denote f*n* (x), F*n* (x) as the *pdf* and *cdf* of the Student's *t*-distribution with degree of freedom *n*, and the *pdf* of the *t*-distribution defined by

$$f_n(x) = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\sqrt{n\pi}}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}, x \in R \tag{7}$$

Following facts from Finner et al [10] will be used later,

$$\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}\frac{1}{\sqrt{n}} \to \frac{1}{2} \quad as \;\; n \to \infty \tag{8}$$

where Γ(·) stands for the complete Gamma function. For *t*-distribution, Soms [12][13] provided the following inequalities:

$$(\frac{1}{x}+\frac{x}{n})(1-\frac{n}{n+2}\frac{1}{x^2}) < \frac{1-F_n(x)}{f_n(x)} < \frac{1}{x}+\frac{x}{n} \text{ for all x > 0,} \tag{9}$$

and:

$$-(\frac{1}{x}+\frac{x}{n})(1-\frac{n}{n+2}\frac{1}{x^2}) < \frac{F_n(x)}{f_n(x)} < -(\frac{1}{x}+\frac{x}{n}) \text{ for all x < 0.} \tag{10}$$

We arrange this paper as follows: In section 2, we consider the asymptotic ratio behavior of *pdf* and *cdf* of AL distribution and *t*-distribution as the degree of freedom of the latter tending to infinity. Similar work has been done for GL distribution and AN distribution in section 3 and 4 respectively. In section 5 and 6 we introduce some applications of asymmetric distributions in the field of financial data modelling and machine learning.

II. Tail Behavior of AL Distribution

For fixed *v, σ, μ*, it is easy to get the following conclusion

$$\frac{f_n(x)}{al_{\sigma,\mu}(x)} \to \infty, \frac{1-F_n(x)}{AL_{\sigma,\mu}(x)} \to \infty, \frac{F_n(-x)}{AL_{\sigma,\mu}(-x)} \to \infty, as \; x \to \infty$$

But under some suitable conditions, we may obtain similar results for AL distribution to those for normal distribution provided in Finner et al. [10].

Firstly we consider the *pdf* ratio and *cdf* ratio of AL distribution and *t*-distribution as $x > 0$.

Theorem 2.1. For the right tail of AL distribution, i.e., $x > 0$, suppose both:

$$n = n(x), \sigma = \sigma(x) = \frac{2n}{(n+1)x}, \mu = \mu(x) = \frac{4n+2}{(n+1)^2 x} \tag{11}$$

and:

$$\lim_{x\to\infty}(x^4 / n - 4\log x) = \beta \in [0,\infty) \tag{12}$$

Then, we have:

$$\lim_{x\to\infty}\frac{f_n(x)}{al_{\sigma,\mu}(x)} = \frac{4}{\sqrt{2\pi}}\exp(\frac{\beta}{4}) \tag{13}$$

Proof. By using (2) and (7), for $x > 0$,

$$\frac{f_n(x)}{al_{\sigma,\mu}(x)} = \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\sqrt{n\pi}}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}\frac{\sigma(1+\kappa^2)}{\kappa}\exp(\frac{\kappa x}{\sigma}) \tag{14}$$

Denote $h_{n,\sigma,\mu}(x) = (\sigma(1+k^2)/\kappa)(1+x^2/n)^{-(n+1)/2}\exp((k/\sigma)x)$, for sufficiently large x, we obtain:

$$\begin{aligned}\log(h_{n,\sigma,\mu}(x)) &= \log\frac{\sigma(1+k^2)}{\kappa} + \frac{\kappa x}{\sigma} - \frac{n+1}{2}\log(1+\frac{x^2}{n}) \\ &= \log\frac{\sigma}{\kappa} + \log(1+k^2) + \frac{\kappa x}{\sigma} - \frac{n+1}{2}(\frac{x^2}{n} - \frac{x^4}{2n^2} + O(\frac{x^6}{n^3})) \\ &= \log\frac{\sigma}{\kappa} + \log(1+k^2) + \frac{\kappa x}{\sigma} - \frac{x^2}{2} - \frac{x^2}{2n} + \frac{n+1}{4n}\frac{x^4}{n} + O(\frac{x^6}{n^2}) \\ &= \log\left(2(1+(\frac{n}{n+1})^2)\right) + \left(\frac{n+1}{4n}\frac{x^4}{n} - \log n\right) - \frac{x^2}{2n} + O(\frac{x^6}{n^2}) \\ &= \log 4 + \frac{\beta}{4} + o(1)\end{aligned}$$

by conditions (11) and (12). Hence by using (8), we complete the proof.

Theorem 2.2. Under the conditions of Theorem 2.1, we have:

$$\lim_{x\to\infty}\frac{1-F_n(x)}{1-AL_{\sigma,\mu}(x)} = \frac{2}{\sqrt{2\pi}}\exp\left(\frac{\beta}{4}\right) \tag{15}$$

Proof. For all $x > 0$, conditions (11) and (12) imply

$$\frac{al_{\sigma,\mu}(x)}{1-AL_{\sigma,\mu}(x)} = \frac{x}{2} \tag{16}$$

Notice

$$\frac{1-F_n(x)}{1-AL_{\sigma,\mu}(x)} = \frac{1-F_n(x)}{f_n(x)}\frac{f_n(x)}{al_{\sigma,\mu}(x)}\frac{al_{\sigma,\mu}(x)}{1-AL_{\sigma,\mu}(x)} \tag{17}$$

and (9) guarantees following inequalities:

$$(1+\frac{x^2}{n})(1-\frac{n}{n+2}\frac{1}{x^2})\frac{f_n(x)}{2al_{\sigma,\mu}(x)} < \frac{1-F_n(x)}{1-AL_{\sigma,\mu}(x)} < (1+\frac{x^2}{n})\frac{f_n(x)}{2al_{\sigma,\mu}(x)} \tag{18}$$

By using Theorem 2.1, the result follows.

An alternative of (16) is the Mills-type ratio of AL distribution.

Corollary 2.1. Under the conditions of Theorem 2.1, the following formula holds

$$\frac{1-AL_{\sigma,\mu}(x)}{al_{\sigma,\mu}(x)} = \frac{2}{x} \tag{19}$$

Similarly we may obtain the left tail ratio behavior of AL distribution and t-distribution. We only list the results and the proofs are omitted.

Theorem 2.3. For the left tail of AL distribution, i.e., $x < 0$, let $n = n(x)$, $\sigma = \sigma(x) = -\frac{2n}{(n+1)x}, \mu = \mu(x) = \frac{4n+2}{(n+1)^2 x}$. Assume that

$$\lim_{x\to-\infty}\left(x^4/n - 4\log(-x)\right) = \beta \in [0,\infty) \tag{20}$$

holds, then

$$\lim_{x\to-\infty}\frac{f_n(x)}{al_{\sigma,\mu}(x)} = \frac{4}{\sqrt{2\pi}}\exp(\frac{\beta}{4}) \tag{21}$$

Theorem 2.4. Under the conditions of Theorem 2.3, we may get

$$\lim_{x\to-\infty}\frac{F_n(x)}{AL_{\sigma,\mu}(x)}=\frac{2}{\sqrt{2\pi}}\exp\left(\frac{\beta}{4}\right) \tag{22}$$

Corollary 2.2. Under the conditions of Theorem 2.3, the following formula holds

$$\frac{AL_{\sigma,\mu}(x)}{al_{\sigma,\mu}(x)}=-\frac{2}{x} \tag{23}$$

That's what we discussed the *pdf* and *cdf* tail ratio behavior of AL distribution and *t*-distribution.

III. Tail Behavior of GL Distribution

In the follwing part, we will consider the asymptotic *pdf* and *cdf* ratio behavior of GL distribution and *t*-distribution as the degree of freedom of the latter tending to infinity.

For the right tail ratio asymptotic behavior, we have following results.

Theorem 3.1. For the right tail of GL distribution, i.e., $x>0$, let $n=n(x)$, $\sigma=\sigma(x)=\frac{2n}{(n+1)x}$. For fixed $b > 0$, if:

$$\lim_{x\to-\infty}\left(x^4/n-4\log(-x)\right)=\beta\in[0,\infty) \tag{24}$$

holds, we have:

$$\lim_{x\to\infty}\frac{f_n(x)}{gl_{b,\sigma}(x)}=\frac{2}{b\sqrt{2\pi}}\exp(\frac{\beta}{4}) \tag{25}$$

Proof. Notice

$$\frac{f_n(x)}{gl_{b,\sigma}(x)}=\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})\sqrt{n\pi}}\left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}\frac{\sigma}{b}\left(1+\exp(-\frac{x}{\sigma})\right)^{b+1}\exp(\frac{x}{\sigma}) \tag{26}$$

Let $k_{n,b,\sigma}(x)=\sigma\exp(x/\sigma)\left(1+\exp(-x/\sigma)\right)^{b+1}(1+x^2/n)^{-(n+1)/2}$ using the similar arguments in Theorem 2.1 and combining with (12), we may get $k_{n,b,\sigma}(x)\to 2\exp(\beta/4)$ as $x\to\infty$. Combining with (8) to complete the proof.

For the asymptotic behavior of *t*-distribution and generalized Logistic distribution, the right tail large deviation theorem may be derived.

Theorem 3.2. Under the conditions of Theorem 3.1above, we get the following conclusion:

$$\lim_{x\to\infty}\frac{1-F_n(x)}{1-GL_{b,\sigma}(x)}=\frac{1}{b\sqrt{2\pi}}\exp\left(\frac{\beta}{4}\right) \tag{27}$$

Proof. Note that under the condition of (24), the following formula holds

$$\frac{gl_{b,\sigma}(x)}{1-GL_{b,\sigma}(x)}=\frac{x}{2}s_n(x) \tag{28}$$

where:

$$s_n(x)=\frac{n+1}{n}\left(1+\exp(-\frac{n+1}{2n}x^2)\right)^{-b-1}\left(1+O(\exp(-\frac{n+1}{2n}x^2))\right)^{-1} \tag{29}$$

with $\lim_{x\to\infty}s_n(x)=1$, notice:

$$\frac{1-F_n(x)}{1-GL_{b,\sigma}(x)}=\frac{1-F_n(x)}{f_n(x)}\frac{f_n(x)}{gl_{b,\sigma}(x)}\frac{gl_{b,\sigma}(x)}{1-GL_{b,\sigma}(x)} \tag{30}$$

By using inequality (9) and formula (28), then:

$$(1+\frac{x^2}{n})(1-\frac{n}{n+2}\frac{1}{x^2})\frac{s_n(x)}{2}\frac{f_n(x)}{gl_{b,\sigma}(x)}<\frac{1-F_n(x)}{1-GL_{b,\sigma}(x)}<(1+\frac{x^2}{n})\frac{s_n(x)}{2}\frac{f_n(x)}{gl_{b,\sigma}(x)} \tag{31}$$

Hence, the inequality of (31) follows by Theorem 3.1

From the proof of Theorem 3.2, we may get the following Mills-type ratio of GL distribution which is similar to Corollary 2.1.

Corollary 3.1. Under the conditions of Theorem 3.1 above, we get the following conclusion

$$\frac{1-GL_{b,\sigma}(x)}{gl_{b,\sigma}(x)} \sim \frac{2}{x} \quad as \quad x \to \infty$$

The left tail ratio behavior of GL distribution and t-distribution may also be obtained by using similar arguments. We also only list the results and omit the proofs.

Theorem 3.3. For the left tail of GL distribution, i.e., $x<0$, let $n=n(x)$, $\sigma=\sigma(x)=-\frac{2n}{n+1}\frac{b}{x}$. Furthermore, suppose:

$$\lim_{x\to-\infty}\left(x^4/n-4\log(-x)\right)=\beta\in[0,\infty) \tag{32}$$

holds, for any fixed $b > 0$, we may have:

$$\lim_{x\to-\infty}\frac{f_n(x)}{gl_{b,\sigma}(x)}=\frac{2}{\sqrt{2\pi}}\exp(\frac{\beta}{4}) \tag{33}$$

Theorem 3.4. Under the conditions of Theorem 3.3, then:

$$\lim_{x\to-\infty}\frac{F_n(x)}{GL_{b,\sigma}(x)}=\frac{1}{\sqrt{2\pi}}\exp\left(\frac{\beta}{4}\right) \tag{34}$$

Corollary 3.2. Under the conditions of Theorem 3.3 above, we get the following conclusion:

$$\frac{GL_{b,\sigma}(x)}{gl_{b,\sigma}(x)} \sim -\frac{2}{x} \quad as \quad x \to \infty$$

We've got the tail ratio behavior of GL distribution and *t*-distribution, and a Mills-type ratio of GL distribution.

IV. Tail behavior of AN distribution

In the following section, we will extend some results in Finner et al. [10] to the AN distribution. For all $x > 0$ the following inequalities and Mills' ratio are well known:

$$\frac{\phi(x)}{x(1+x^{-2})}<1-\Phi(x)<\frac{\phi(x)}{x} \tag{35}$$

and,

$$\frac{1-\Phi(x)}{\phi(x)} \sim \frac{1}{x} \quad as \quad n \to \infty \tag{36}$$

Let $an_{\theta,\sigma_l,\sigma_r}(x)$ stands for the density of AN distribution. We may get following inequalities and Mills-type ratio alike to (35) and (36) for the AN distribution.

Lemma 4.1. For any $\sigma_l,\sigma_r>0,$ and θ ∈ R, then:

$$\frac{{\sigma_l}^2}{\theta-x}\left(1+(\frac{{\sigma_l}^2}{\theta-x})^2\right)^{-1}<\frac{AN_{\theta,\sigma_l,\sigma_r}(x)}{an_{\theta,\sigma_l,\sigma_r}(x)}<\frac{{\sigma_l}^2}{\theta-x} \quad x\le\theta, \tag{37}$$

$$\frac{{\sigma_r}^2}{\theta-x}\left(1+(\frac{{\sigma_r}^2}{x-\theta})^2\right)^{-1}<\frac{1-AN_{\theta,\sigma_l,\sigma_r}(x)}{an_{\theta,\sigma_l,\sigma_r}(x)}<\frac{{\sigma_r}^2}{\theta-x} \quad x>\theta, \tag{38}$$

Proof: The proofs of the two statements are the same in principle. We only prove (i).

For any $\sigma_l,\sigma_r>0, x\le\theta$,the *cdf* of AN distribution is:

$$AN_{\theta,\sigma_l,\sigma_r}(x)=\frac{2\sigma_l}{\sqrt{2\pi}(\sigma_l+\sigma_r)}\int_{\frac{\theta-x}{\sigma_l}}^{+\infty}\exp(-\frac{t^2}{2})dt \tag{39}$$

By using lemma 3 in Azzalini and Dalla[8], we will get some inequalities

$$\frac{\sigma_l}{\theta-x}\left(1+(\frac{\sigma_l^{\,2}}{x-\theta})^2\right)^{-1}\exp\left\{-\frac{1}{2}(\frac{x-\theta}{\sigma_l})^2\right\}<\int_{\frac{\theta-x}{\sigma_l}}^{+\infty}\exp(-\frac{t^2}{2})dt$$

$$<\frac{\sigma_l}{\theta-x}\exp\left\{-\frac{1}{2}(\frac{x-\theta}{\sigma_l})^2\right\} \tag{40}$$

The result follows.

Theorem 4.1. For any $\sigma_l,\sigma_r>0,$ and θ ∈ R, we have the following Mills-type ratios:

$$\frac{AN_{\theta,\sigma_l,\sigma_r}(x)}{an_{\theta,\sigma_l,\sigma_r}(x)}\sim\frac{\sigma_l^{\,2}}{x}\quad as\quad x\to-\infty \tag{41}$$

and,

$$\frac{1-AN_{\theta,\sigma_l,\sigma_r}(x)}{an_{\theta,\sigma_l,\sigma_r}(x)}\sim\frac{\sigma_r^{\,2}}{x}\quad as\quad x\to\infty \tag{42}$$

Now let $n = n(x)$, the degree of freedom of *t*-distribution, such that

$$\lim_{x\to-\infty}\frac{x^4}{n}=\beta\in[0,\infty) \tag{43}$$

The following asymptotic ratio behaviors of *pdf* and *cdf* of AN and *t*-distribution may be derived.

Theorem 4.2. For fxed $\sigma_l,\sigma_r>0,\theta\in R$, and moreover, let $n = n(x)$ and (43) holds. Then:

$$\lim_{x\to-\infty}\frac{f_n(x)}{an_{\theta,\sigma_l,\sigma_r}(\sigma_l x+\theta)}=\frac{\sigma_l+\sigma_r}{2}\exp(\frac{\beta}{4}) \tag{44}$$

and

$$\lim_{x\to\infty}\frac{f_n(x)}{an_{\theta,\sigma_l,\sigma_r}(\sigma_r x+\theta)}=\frac{\sigma_l+\sigma_r}{2}\exp(\frac{\beta}{4}) \tag{45}$$

Proof: The proofs are very similar to Theorem 1.1 in Finner et al. (2008) [10], which are omitted here.

Theorem 4.3. Under the conditions of Theorem 4.2 above, we get the following conclusion

$$\lim_{x\to-\infty}\frac{F_n(x)}{AN_{\theta,\sigma_l,\sigma_r}(\sigma_l x+\theta)}=\frac{\sigma_l+\sigma_r}{2\sigma_l}\exp(\frac{\beta}{4}) \tag{46}$$

and

$$\lim_{x\to\infty}\frac{1-F_n(x)}{1-AN_{\theta,\sigma_l,\sigma_r}(\sigma_r x+\theta)}=\frac{\sigma_l+\sigma_r}{2\sigma_r}\exp(\frac{\beta}{4}) \tag{47}$$

Proof: Here, we only consider the left tail ratio asymptotic behavior. Let $\theta^*=\min\{0,\theta\}$, for all $x\le\theta^*,n>0$, then

$$\frac{F_n(x)}{AN_{\theta,\sigma_l,\sigma_r}(\sigma_l x+\theta)}=\frac{F_n(x)}{f_n(x)}\frac{f_n(x)}{an_{\theta,\sigma_l,\sigma_r}(\sigma_l x+\theta)}\frac{an_{\theta,\sigma_l,\sigma_r}(\sigma_l x+\theta)}{AN_{\theta,\sigma_l,\sigma_r}(\sigma_l x+\theta)} \tag{48}$$

By using (10), (43), Lemma 4.1 and Theorem 4.2, we get the desired results.

For the tail characteristics of α-partial (α≠0) normal distribution, a Mills-type ratio was given [14]. This section builds on this for the more general Mills-type ratio of AN.

V. Applications of Asymmetric Distributions

Asymmetric distributions have a wide range of applications in the field of financial data modelling. The asymmetric Laplace distribution fit the stock market returns much better [15]. The asymmetric Laplace distribution and asymmetric normal distribution can describe the peak-fat tail bias better than normal distribution, and has a good application value in financial market data modeling [16-17]. Kozubowski used asymmetric Laplace, normal and stable Pareto distributions to model the currency exchange rate data, both asymmetric Laplace and stable Pareto distributions can better describe the spike thick-tailed bias characteristics, portray abnormal spikes around the mean, and the asymmetric Laplace distribution reduces the computational time and resource consumption [11].Jayakumar and Kuttykrishnan took the AL distribution as the marginal distribution , established a time-series model when they processed and analyzed financial time-series data [18] . In the papers of Kozubowski and Podgorski [19][20], they studied the application of AL distribution in modeling currency exchange rate and interest rate. The result showed that AL distribution has the same properties as normal distribution in some respects, and also appears to be an alternative to stable Paretian distributions. Wu et al. constructed a class of base semiparametric asymmetric joint derivable risk models under the assumption that asset returns obey an asymmetric Laplace distribution, which has a more prominent joint risk modelling and prediction performance [21].Ina Huang derived the generalised skewed logistic distribution by introducing the skew parameter, which can fit the distribution of the logarithmic return of China's SSE index very well, and in the case of a very small probability, the distribution can estimate a large risk [22].Wen Luliang et al. used an asymmetric generalised normal distribution for the daily return dataset of the Standard & Poor's 500 Index and the Shanghai Stock Exchange Composite Index, which was fitted with an asymmetric generalised normal distribution, and this distribution was able to fit the spike-thick-tailed band skewed characteristics of the daily return data better [23].Further applications of asymmetric distributions in the field of financial data modelling could be found in Liu [24], Eruore [25], and Yu [26].

Asymmetric distributions are widely used in the field of machine learning. Li et al. constructed a three-parameter generalized logistic distribution model and modelled the cyclone grading data, which has a high fitting accuracy and is a well-adapted and parametrically meaningful mathematical model of the grading curve [27]. Zhang et al. modelled the performance degradation data based on the type II generalized logistic distribution, which can effectively mine the tail characteristics of the degradation data, truly reflect the initial degradation of the product, and accurately express the trend of the degradation data over time [28]. Mameli proposed the Kumaraswamy skewed normal distribution and used this distribution to analyse a dataset of body mass index measurements of Australian athletes, which was single-peaked and easy to handle compared to the BSN distribution [29]. Gan et al. proposed a new three-parameter asymmetric normal distribution by introducing a skew-tailed parameter in the normal distribution, which has a good fitting effect on both the Danish fire loss data and the body mass index data, and can reasonably reflect the skewed characteristics existing in the data distribution [30].

VI. Conclusions

In the extreme value theory, the research on the concrete distribution is more and more extensive. For the study of the tail characteristics of distribution functions, Mills' ratio plays an important role .We only deals with the *pdf* and *cdf* tail ratio behavior of three asymmetric distributions and t-distribution, and get their Mills' ratio in this paper. Asymmetric distributions play a key role in various fields such as financial data modelling and machine learning.

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