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Hermite –Hadamard and Hermaite –Hadamard Fejer inequality for new f-divergence measure via conformable fractional integrals

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Abstract: -. In information geometry, a divergence measure is a type of statistical distance and a binary function which established the separation from one probability distribution to another on a statistical numerous. The basic use of divergence measure is statistical data processing, information storage, decision making etc. The most famous inequality regarding the integral mean of a convex function is HermiteâHadamard's inequality and the weighted version of this is called the Hermite-Hadamard-Fejér inequality. Purpose of this paper is to find Hermaite-Hadamard and Hermite-Hadamard Fejer type inequalities for new f-divergence measure with the help of conformable fractional integrals. Hermaite-Hadamard inequality gives us necessary and sufficient condition for a function must be convex. Here we consider the new f-divergence measure, it has the property of convexity. In this research article we drive some inequalities for t-convex function which gives us the extensions of the previous work for convex and t-convex function and also obtains some fractional midpoint type inequalities. The main purpose of this paper is to establish conformal fractional approximation of Hermaite-Hadamard and Hermite-Hadamard Fejer type inequalities for new f-divergence measure which close the fractional integral and the Riemann-Liouville integrable into single form,also gives us some new results for ψ -Riemann-Liouville integral as special cases of main results. This article gives us most useful link between convexity and symmetry.

Keywords: Hermite-Hadamard inequality, Hermite-Hadamard Fejer type inequality, Riemann-Liouville fractional integral, new f-divergence measure, t-convex function, conformable fraction integral.

I.INTRODUCTION

In the literature of information theory inequalities play a useful role in finding the relations between divergence measures, bounds, coding, and various field. Various mathematicians have used various types of inequalities. The Hermite-Hadamard inequality is one of the most significant inequality. Initially it was discovered by Hermite and afterward by Hadamard. Here , giving some examples of these type of inequalities

Fractional Hermite-Hadamard-Fejer inequalities for a convex function with respect to an increasing function [1]. New extension of Hermite –Hadamard inequalities for generalized fractional integrals [2]. Improvement of fractional Hermite-Hadamard type inequality for conves function [3]. Hermite-Hadamard-fejer type inequalities for p-convex function via fractional integral [4]. New inequalities on Fejer and Hermite-Hadamard type inequalities involving h-convex function and applications [5]. Hermite-Hadamard-fejer type fractional inequalies for convex function [6]. Hermite-Hadamard-fejer type inequalities for the p-convex function via α-generator [7].In the present article we will present some HH and HH-F inequalities for new f-divergence measure with the support of conformable fractional integral.

A function η defined $M \to R$ and $m_1, m_2 \in M$, $l \in [0,1]$ is known as convex if the relation holds.

$$\eta\left(lm_1 + (1-l)m_2\right) \le l\eta(m_1) + (1-l)\eta(m_2) \tag{1}$$

Many authors have given the importance of convex functions and their generalizations of the convex functions. The Hermite-Hadamard inequality defined in the interval.

$$\eta\left(\frac{m_1+m_2}{2}\right) \le \frac{1}{(m_2-m_1)}\int_{m_1}^{m_2}\eta(s)\,ds \le \frac{\eta(m_1)+\eta(m_2)}{2} \tag{2}$$

Here the function η is nonnegative symmetric and integrable also. to $\frac{m_1+m_2}{2}$, known as Hermite-Hadamard-Fejer inequality. Further generalization of the inequality (2) and (3) in the different ways not only classical integral $\forall m_1, m_2 \in M$ With $m_1 < m_2$. Then the following inequality proved by Fejer. [8].

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$$\eta\left(\frac{m_1+m_2}{2}\right)\int_{m_1}^{m_2}\eta(s)\,ds \le \frac{1}{m_2-m_1}\int_{m_1}^{m_2}\eta(s)\eta(s)\,ds \le \frac{\eta(m_1)+\eta(m_2)}{2}\int_{m_1}^{m_2}\eta(s)\,ds \qquad (3)$$

but also other generalized as Riemann-Liouville integrable, ψ -Riemann-Liouville and conformable fractional integral etc.

Definition1 [8, 9]

Let the interval $M \subset (0,\infty)$ and $t \in R\setminus\{0\}$. Then the function $\eta : M \to R$ called t-convex if

$$\eta\left(\left[lm_1^t+(1-l)m_2^t\right]^{1/t} \le l\eta(m_1)+(1-l)\eta(m_2)\right) \qquad (4)$$

$$\forall m_1, m_2 \in M \text{ And } l \in [0,1]$$

Definition2 [10]

Let $\eta \in L(m_1,m_2)$ the left and right sided in the Reiman-Liouville fraction integrable $J^{o}_{m_1^+}\eta$ and $J^{o}_{m_1^-}\eta$ of order $o \in c$ with $R(o)$ >0 and $m_2 > m_2 \ge 0$ are given by

$$J^{o}_{m_1^+}\eta(s) = \frac{1}{\Gamma(\alpha)}\int_{m_1}^{s}(s-w)^{o-1}\eta(w)\,dw, \qquad s > m_1$$

And

$$J^{o}_{m_2}\eta(s) = \frac{1}{\Gamma(\alpha)}\int_{s}^{m_2}(w-s)^{o-1}\eta(w)\,dw, \qquad s < m_2$$

Respectively, where Γ (.) be known as gamma function. The conformal fractional integral as follows.

Definition3 [11]

Let $o \in (n, n+1]$ and $\gamma = o-n$ .then the left and right sided conformable fractional integrals of order $o > 0$ are given by

$$J^{m_1}_{o}\eta(s) = \frac{1}{\lfloor n}\int_{m_1}^{s}(s-w)^{n}(w-m_1)^{\eta-1}\eta(w)\,dw$$

And

$$J^{m_2}_{o}\eta(s) = \frac{1}{\lfloor n}\int_{s}^{m_2}(w-s)^{n}(m_2-w)^{\eta-1}\eta(w)\,dw$$

Respectively

$\Rightarrow$ For $o = n+1$ then $\eta = 1$

Where $n = 0,1,2,\ldots\ldots\ldots$ and in this case conformable fraction integrals becomes Riemann- Liouville fractional integrals. The classical beta function and hypergeometric function are defined, respectively, by

$$\eta(m_1,m_2) = \int_{0}^{1} w^{m_1-1}(1-w)^{m_2-1}\,ds$$

and

$${}_2F_1(m_1,m_2;s,w) = \frac{1}{\beta'(m_2,s-m_2)}\int_{0}^{1} s^{m_2-1}(1-s)^{s-m_2-1}(1-sw)^{-m_1}\,ds$$

With $s > m_2 > 0, |w| < 1$.

This is known as incomplete beta function.

$$\beta'_s(m_1,m_2) = \int_{0}^{u} w^{m_1-1}(1-w)^{m_2-1}\,dw \qquad , \quad w \in [0,1]$$

The classical beta function and the incomplete beta function is given as follows.

$$\beta'(m_1,m_2) = \beta'_s(m_1,m_2) + \beta'_{1-s}(m_1,m_2)$$

Definition4 [12] New f-divergence measure and its properties is given by

$$S_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i+q_i}{2q_i}\right)$$

A function f defined as $[0,\infty)\to R$ is the convex, then we have the following inequality $S_f(P,Q)\geq f(1)$

II.HERMITE-HADAMARD INEQUALITIES

In this section, Established Hermite-Hadamard inequalities for new f-divergence measure with the help of conformable fraction integral mapping.

Theorem1 [13]
Let $\eta:[m_1,m_2]\subset(0,\infty)\to R$ (here $\eta=s_f(p,q)$) is a t-convex function such that $\eta\in L[m_1,m_2]$ and $o>0$. Then
i.For $t>0$, we have

$$m_2\eta\left(\left[\frac{m_1^t+m_2^t}{2m_2^t}\right]^{1/t}\right)\leq\frac{\Gamma o+1}{2\Gamma o-n}\frac{m_2}{\left(m_2^t-m_1^t\right)}\left[J_o^{m_1^t}\left(\eta(\phi)\right)m_2^t+J_o^{m_2^t}\left(\eta(\phi)m_1^t\right)\right]\leq\frac{\eta\left(m_1^t\right)+\eta\left(m_2^t\right)}{2} \tag{5}$$

Here $\varphi(s)=s^{1/t}\,\forall s\in\left[m_1^t,m_2^t\right]$
ii.For $t<0$,we have

$$m_2\eta\left(\left[\frac{m_1^t+m_2^t}{2m_2^t}\right]^{1/t}\right)\leq\frac{\Gamma o+1}{2\Gamma o-n}\frac{m_2}{\left(m_1^t-m_2^t\right)}\left[J_o^{m_1^t}\left(\eta(\phi)\right)m_2^t+J_o^{m_2^t}\left(\eta(\phi)m_1^t\right)\right]\leq\frac{\eta\left(m_1^t\right)+\eta\left(m_2^t\right)}{2}$$

(6)

Here $\varphi(s)=s^{1/t}\,\forall s\in\left[m_2^t,m_1^t\right]$

Proof
η be a t-convex function on $[m_1,m_2]$, we have

$$b\eta\left(\left[\frac{a^t+b^t}{2b^t}\right]^{1/t}\right)\leq\left(\frac{\eta(a)+\eta(b)}{2}\right)$$

Now take $a^t=lm_1^t+(1-l)m_2^t$ and $b^t=(1-l)m_1^t+lm_2^t$ with $l\in[0,1]$ then we get

$$m_2\eta\left(\left[\frac{m_1^t+m_2^t}{2m_2^t}\right]^{1/t}\right)\leq\frac{\eta\left[lm_1^t+(1-l)m_2^t\right]^{1/t}+\eta\left[(1-l)m_1^t+lm_2^t\right]^{1/t}}{2} \tag{7}$$

Multiplying with $\frac{1}{\lfloor n}l^n(1-l)^{o-n-1}$ equation (7), Here $l\in(0,1), o>0$, on both sides and integrating about k over $[0,1]$ then we have

$$\frac{2m_2}{\lfloor n}\eta\left(\left[\frac{m_1^t+m_2^t}{2}\right]^{1/t}\right)\int_0^1 l^n(1-l)^{o-n-1}\,dl\leq\frac{1}{\lfloor n}\int_0^1 l^n(1-l)^{o-n-1}\eta\left(\left[lm_1^t+(1-l)m_2^t\right]^{1/t}\right)dl$$

$$+\frac{1}{\lfloor n}\int_0^1 l^n(1-l)^{o-n-1}\eta\left(\left[(1-l)m_1^t+lm_2^t\right]^{1/t}\right)=I_1^{;}+I_2^{;} \tag{8}$$

By putting the value $s=lm_1^t+(1-l)m_2^t$ we have

$$I_1^{;}=\frac{1}{\lfloor n}\int_0^1 l^n(1-l)^{o-n-1}\eta\left(\left[lm_1^t+(1-l)m_2^t\right]^{1/t}\right)dl$$

$$=\frac{1}{\lfloor n}\int_{m_2^t}^{m_1^t}\left(\frac{s-m_2^t}{m_1^t-m_2^t}\right)^n\left(1-\frac{s-m_2^t}{m_1^t-m_2^t}\right)^{o-n-1}\left(\eta(\phi)\right)\frac{ds}{m_1^t-m_2^t}$$

$$=\frac{1}{\lfloor n}\frac{1}{\left(m_2^t-m_1^t\right)^o}\int_{m_1^t}^{m_2^t}\left(m_2^t-s\right)^n\left(s-m_1^t\right)^{o-n-1}\left(\eta(\phi)\right)ds$$

$$= \frac{1}{\left(m_2^t - m_1^t\right)^o} J_o^{m_1^t}\left(\eta(\phi)\right)\left(m_2^t\right) \quad (9)$$

By putting $s = lm_2^t + (1-l)m_1^t$, we have

$$I_2^{\cdot} = \frac{1}{\lfloor n}\int_0^1 l^n (1-l)^{o-n-1}\eta\left(\left[lm_2^t + (1-l)m_1^t\right]^{1/t}\right)dl$$

$$= \frac{1}{\lfloor n}\int_{m_1^t}^{m_2^t}\left(\frac{s-m_1^t}{m_2^t - m_1^t}\right)\left(1-\frac{u-m_1^t}{m_2^t - m_1^t}\right)^{o-n-1}\left(\eta(\phi(s))\right)\frac{ds}{m_2^t - m_1^t}$$

$$= \frac{1}{\lfloor n\left(m_2^t - m_1^t\right)^o}\int_{m_1^t}^{m_2^t}\left(s-m_1^t\right)\left(m_2^t - u\right)^{o-n-1}\left(\eta(\phi(s))\right)ds$$

$$= \frac{1}{\left(m_2^t - m_1^t\right)^o} {}_{m_2^t}J_o\left(\eta(\phi(s))\right)\left(m_1^t\right) \quad (10)$$

Now putting the value of $I_1^{'}$ and $I_2^{\cdot}$ in the equation (9), the first inequality of (5) is prove, for other inequality, we note that,

$$\eta\left(\left[lm_1^t + (1-l)m_2^t\right]^{1/t}\right) + \eta\left(\left[lm_2^t + (1-l)m_1^t\right]^{1/t}\right) \le \left[\eta(m_1) + \eta(m_2)\right] \quad (11)$$

Multiplying with $\frac{1}{\lfloor n} l^n (1-l)^{o-n-1}$, here $l \in (0,1), o > 0$, on both sides and then integrating about l over $[0,1]$, hence the inequality (11). This completes the proof.

Lemma1

Suppose $\eta : [m_1, m_2] \subset (0,\infty) \to R$ be a differentiable function on (m_1, m_2) with $m_1 < m_2$ such that,

I.For $t > 0$, we have

$${}_1\Delta_\eta(m_1, m_2; \beta; J) = m_2\left(\frac{m_2^t - m_1^t}{2tm_2^t}\right)\int_0^1 \beta_{1-l}(n+1, o-n) - \beta_l(n+1, o-n) \times A_l^{1/t}\eta^{\cdot}\left(\left[lm_1^t + (1-l)m_2^t\right]^{1/t}\right)dl \quad (12)$$

Here $A_l = \left[lm_1^t + (1-l)m_2^t\right]$ and ${}_1\Delta_\eta(m_1, m_2; o; \beta; J)$

$$= \beta(n+1, o-n)\left(\frac{\eta(m_1)+\eta(m_2)}{2}\right) - \frac{\lfloor n m_2^t}{2\left(m_2^t - m_1^t\right)^o}\left[J_o^{m_1^t}\left(\eta\left(\phi\left(m_2^t\right)\right)\right) + J_o^{m_2^t}\eta\left(\phi\left(\begin{matrix}1\\ t\end{matrix}\right)\right)\right]$$

II.For $t < 0$, we have

$${}_2\Delta_\eta(m_1, m_2; \beta; J; 1) = m_2\left(\frac{m_2^t - m_1^t}{2tm_2^t}\right)\int_0^1 \beta_l(n+1, o-n) - \beta_{1-l}(n+1, o-n) \times B_l^{1/t}\eta^{\cdot}\left(\left[lm_2^t + (1-l)m_1^t\right]^{1/t}\right)dl \quad (13)$$

Here $B_l = \left[lm_2^t + (1-l)m_1^t\right]$ and ${}_2\Delta_\eta(m_1, m_2; o; \beta; J) = \beta(n+1, o-n)\left(\frac{\eta(m_1)+\eta(m_2)}{2}\right)$

$$-\frac{\lfloor n m_2^t}{2\left(m_1^t - m_2^t\right)^o}\left[J_o^{m_1^t}\left(\eta\left(\phi\left(m_2^t\right)\right)\right) + J_o^{m_2^t}\left(\eta\left(\phi\left(m_1^t\right)\right)\right)\right]$$

Proof

consider $\int_0^1 \beta_{1-l}^{'}(n+1, o-n) - \beta_l^{\cdot}(n+1, o-n) A_l^{\frac{1}{t}-1}\eta^{\cdot}\left[lm_1^t + (1-l)m_2^t\right]^{1/t} dl$

$$= \int_0^1 \beta_{1-l}^{\cdot}(n+1, o-n) A_l^{\frac{1}{t}-1}\eta^{\cdot}[\left(lm_1^t + (1-l)m_2^t\right)^{1/t}]dl - \int_0^1 \beta_l^{\cdot}(n+1, o-n) A_l^{\frac{1}{t}-1}\eta^{\cdot}[\left(lm_1^t + (1-l)m_2^t\right)^{1/t}]dl$$

$$= I_1^{\cdot} - I_2^{\cdot} \quad (14)$$

Using by part integration then we have

$$I_1^{'} = \int_0^1 \beta_{1-l}^{'}(n+1,o-n) - \beta_l^{'}(n+1,o-n) A_l^{\frac{1}{t}-1} \eta^{'}\left[lm_1^t + (1-l)m_2^t\right]^{1/t} dl$$

$$= \int_0^1 \left(\int_0^{1-l} s^n (1-s)^{o-n-1} ds\right) A_l^{\frac{1}{t}-1} \eta^{'}\left(\left[lm_1^t + (1-l)m_2^t\right]^{1/t}\right) dl$$

$$= \frac{tm_2^t}{m_2^t - m_1^t}\beta^{'}(n+1,o-n)\eta(m_2) - \frac{lm_2^t}{m_2^t - m_1^t}\int_0^1 (1-l)^n l^{o-n-q} \eta\left(\left[lm_1^t + (1-l)m_2^t\right]^{1/t}\right) dl$$

$$= \frac{tm_2^t}{m_2^t - m_1^t}\beta(n+1,o-n)\eta(m_2) - \frac{tm_2^t}{m_2^t - m_1^t}\int_{m_2^t}^{m_1^t}\left(1 - \frac{a - m_2^t}{m_1^t - m_2^t}\right)^n \left(\frac{a - m_2^t}{m_1^t - m_2^t}\right)^{o-n-1} \frac{\eta(\phi(x))}{m_1^t - m_2^t} dx$$

$$= \frac{tm_2^t}{m_2^t - m_1^t}\beta(n+1,o-n)\eta(m_2) - \frac{\lfloor nm_2^t}{\left(m_2^t - m_1^t\right)^{o+1}} J_o^{m_2^t}\eta\left(\phi\left(m_1^t\right)\right) \tag{15}$$

Similarly, we have

$$I_2^{'} = \int_0^1 \beta_l^{'}(n+1,o-n)\eta^{'}\left[lm_1^t + (1-l)m_2^t\right]^{1/t} dl$$

$$= \int_0^1 \left(\int_0^l s^n (1-s)^{o-n-1} ds\right)\eta^{'}\left(\left[lm_1^t + (1-l)m_2^t\right]^{1/t}\right) dl$$

$$-\frac{tm_2^t}{m_2^t - m_1^t}\beta^{'}(n+1,o-n)\eta(m_1) + \frac{tm_2^t}{m_2^t - m_1^t}\int_0^1 l^n (1-l)^{o-n-1}\eta\left(\left[lm_1^t + (1-l)m_2^t\right]^{1/t}\right) dl$$

$$-\frac{tm_2^t}{m_2^t - m_1^t}\beta^{'}(n+1,o-n)\eta(m_1) + \frac{tm_2^t}{m_2^t - m_1^t}\int_{m_2^t}^{m_1^t}\left(\frac{a - m_2^t}{m_1^t - m_2^t}\right)^n \left(1 - \frac{a - m_2^t}{m_1^t - m_2^t}\right)^{o-n-1} \frac{\eta(\phi(x))}{m_1^t - m_2^t} dx \tag{16}$$

Substituting values of $I_1^{'}$ and $I_2^{'}$ in the equation (14) and multiplying with $\frac{m_2^t - m_1^t}{2}$, we get

Second proof is similar to first.

Theorem 2

Let $\eta : [m_1, m_2] \subset (0,\infty) \to R$ be a differentiable function on (m_1, m_2) with $m_1 < m_2$ such that $\eta^{'} \in L[m_1, m_2]$ and $o > 0$ .if $\left|\eta^{'}\right|^q$, where $q \geq 1$ is a convex function, then

i. For $t > 0$, we have

$$\left|{}_1\Delta_\eta(m_1, m_2; o; \beta; J)\right| \leq \frac{m_2^t - m_1^t}{2tm_2^t}\chi^{1-\frac{1}{q}}\left(\chi_1\left|\eta^{'}(m_1)\right|^q + \chi_2\left|\eta^{'}(m_2)\right|^q\right)^{1/q} \tag{17}$$

$$\chi = \beta(n+1,o-n+1) - \beta(n+1,o-n) + \beta(n+2,o-n)$$

$$\chi_1 = \frac{m_2^{1-t}}{2}\,{}_2F_1\left(1-\frac{1}{t},2;3;1-\frac{m_1^t}{m_2^t}\right) \text{ And}$$

$$\chi_2 = \frac{m_2^{1-t}}{2}\,{}_2F_1\left(1-\frac{1}{t},1;3;1-\frac{m_1^t}{m_2^t}\right)$$

ii. For $t < 0$, we have

$$\left|{}_2\Delta_\eta(m_1, m_2; o; \beta; J)\right| \leq \frac{m_1^t - m_2^t}{2tm_2^t}\chi_3^{1-\frac{1}{q}}\left(\chi_4\left|\eta^{'}(m_1)\right|^q + \chi_5\left|\eta^{'}(m_2)\right|^q\right)^{1/q} \tag{18}$$

$$\chi_3 = \beta(n+1,o-n+1) - \beta(n+2,o-n)$$

$$\chi_4 = \frac{m_2^{1-t}}{2}\,{}_2F_1\left(1-\frac{1}{t},1;3;1-\frac{m_2^t}{m_1^t}\right)$$

And

$$\chi_5 = \frac{m_2^{t-1}}{2}\ {}_2F_1\left(1-\frac{1}{t},2;3;1-\frac{m_2^t}{m_1^t}\right)$$

Proof

i. $A_K = \left[lm_1^t + (1-l)m_2^t\right]$ Applying the Lemma 1, power mean inequality and t-convexity of $\left|\eta'\right|$, we find

$$\left|{}_1\Delta_n\left(m_1,m_2;o;\beta;J\right)\right| = \left|\frac{m_2^t-m_1^t}{2tm_2^t}\int_0^1\{\beta_{1-l}\left(n+1,o-n\right)-\beta_l\left(n+1,o-n\right)\}\times A_l^{\frac{1}{t}-1}\eta'\left(\left[lm_1^t+(1-l)m_2^t\right]^{1/t}\right)dl\right|$$

$$\le \frac{m_2^t-m_1^t}{2tm_2^t}\left(\int_0^1\left\{\beta_{1-l}\left(n+1,o-n\right)-\beta_l\left(n+1,o-n\right)\right\}dl\right)^{1-1/q}$$

$$\times\left(\int_0^1 A_l^{\frac{1}{t}-1}\left|\eta'\left[lm_1^t+(1-l)m_2^t\right]^{1/t}\right|^q dl\right)^{1/q}$$

$$\le \frac{m_2^t-m_1^t}{2tm_2^t}\chi^{1-1/q}\left(\int_0^1 A_l^{\frac{1}{t}-1}\left[l\left|\eta'\left(m_1\right)\right|^q+(1-l)\left|\eta'\left(m_2\right)\right|^q\right]^{1/q}dl\right)$$

$$= \frac{m_2^t-m_1^t}{2tm_2^t}\chi^{1-1/q}\left[\chi_1\left|\eta'\left(m_1\right)\right|^q+\chi_2\left|\eta'\left(m_2\right)\right|^q\right]^{1/q} \qquad (19)$$

Where

$$\chi = \int_0^1\left(\beta_{1-l}\left(n+1,o-n\right)-\beta_l\left(n+1,o-n\right)\right)dl$$

$$= \int_0^1\left(\int_0^{1-l} s^n\left(1-s\right)^{o-n-1}ds\right)dl + \int_0^1\left(\int_0^l s^n\left(1-s\right)^{o-n-1}\right)dl$$

$$= s\left(\int_0^{1-l} s^n\left(1-s\right)^{o-n-1}ds\right)\Bigg|^1 + \int_0^1 l\left(1-l\right)^{o-n-1}dl$$

$$+ l\left(\int_0^l s^n\left(1-s\right)^{o-n-1}ds\right)\Bigg|_0^1 + \int_0^1 l^n\left(1-l\right)^{o-n-1}dl$$

$$\beta_{1-l}\left(n+1,o-n+1\right)-\beta_l\left(n+1,o-n\right)+\beta\left(n+2,o-n\right)$$

$$\chi_1 = \int_0^1 lA_l^{\frac{1}{t}-1}dl = \frac{m_2^{1-t}}{2}\ {}_2F_1\left(1-\frac{1}{t},2;3;1-\frac{m_1^t}{m_2^t}\right)$$

And $$\chi_2 = \int_0^1(1-l)A_l^{\frac{1}{t}-1}dl = \frac{m_2^{1-t}}{2}\ {}_2F_1\left(1-\frac{1}{t},1;3;1-\frac{m_1^t}{m_2^t}\right)$$

Hence the proof

Proof (ii) is similar to (i)

Theorem3

Let $\eta:\left[m_1,m_2\right]\subset\left(0,\infty\right)\to R$ is a differentiable function on $\left(m_1,m_2\right)$ with the relation $m_1 < m_2$ such that $\eta' \in L\left(m_1,m_2\right)$ and $o>0$. if $\left|\eta'\right|^q$, where $q>1$ is a t-convex function then for $t>0$,we have

$$\left|{}_1\Delta_2\left(m_1,m_2;o;\beta:J\right)\right| \le \frac{m_2^t-m_1^t}{2tm_2^t}\rho^{1-\frac{1}{q}}\left(\left(\rho_1-\rho_2\right)\left|\eta'\left(m_1\right)^q\right|+\left(\rho_3-\rho_4\right)\left|\eta'\left(m_2\right)^q\right|\right)^{1/q}$$

Here $$\chi = \frac{m_2^{1-t}}{2}\ {}_2F_1\left(1-\frac{1}{t},1;2;1-\frac{m_1^t}{m_2^t}\right)$$

$$\chi_1 = \frac{1}{2}\beta\left(n+1,o-n+2\right)$$

$$\chi_2 = \frac{1}{2}\beta(n+1,o-n) - \beta(n+3,o-n)$$

$$\chi_3 = \frac{1}{2}\beta(n+2,o-n+1) - \frac{1}{2}\beta(n+1,o-n+2)$$

And

$$\chi_4 = \frac{1}{2}\beta(n+1,o-n) + \frac{1}{2}\beta(n+3,o-n) - \beta(n+2,o-n)$$

Proof

Applying the result of Lemma1, power mean inequality and t-convex of $|\eta'|^q$ we have

$$\left|{}_1\Delta_2(m_1,m_2;o;\beta:J)\right| = \left|\frac{m_2^t - m_1^t}{2tm_2^t}\int_0^1 \{\beta_{1-l}(n+1,o-n) - \beta_l(n+1,o-n)\} \times A_l^{1/t-1}\eta'\left(lm_1^t + (1-l)m_2^t\right)^{\frac{1}{t}} dl\right|$$

$$\le \frac{m_2^t - m_1^t}{2tm_2^t}\left(A_l^{1/t-1}dl\right)^{1-\frac{1}{q}} \times \left(\int_0^1 \{\beta_{1-l}(n+1,o-n) - \beta_l(n+1,o-n)\}\left|\eta'\left[lm_1^t + (1-l)m_2^t\right]^{1/t}\right|^q dl\right)^{1/q}$$

$$\le \frac{m_2^t - m_1^t}{2tm_2^t}\chi^{1-\frac{1}{q}}\left(\int_0^1 \{\beta_{1-l}(n+1,o-n) - \beta_l(n+1,o-n)\} \times \left[l\left|\eta'(m_1)\right|^q + (1-l)\left|\eta'(m_2)\right|\right]dl\right)^{1/q}$$

$$\le \frac{m_2^t - m_1^t}{2tm_2^t}\chi^{1-\frac{1}{q}}\left(\left[(\chi_1 - \chi_2)\left|\eta'(m_1)\right|^q + (\chi_3 - \chi_4)\left|\eta'(m_2)\right|^q\right]\right)^{1/q} \qquad (20)$$

Where

$$\chi = \int_0^1 A_l^{\frac{1}{l}-1} = \frac{m_2^{1-t}}{2}\; {}_2F_1\left(1-\frac{1}{t},1;2;1-\frac{m_2^t}{m_1^t}\right)$$

$$\chi_1 = \int_0^1 l\beta_{1-l}(n+1,o-n)dl = \frac{1}{2}\beta(n+1,o-n+2) \quad \chi_2 = \int_0^1 l\beta_l(n+1,o-n) = \frac{1}{2}\left(\beta(n+1,o-n) - \beta(n+3,o-n)\right)$$

$$\chi_3 = \int_0^1 (1-l)\beta_{1-l}(n+1,o-n) = \beta(n+2,o-n+1) - \frac{1}{2}\beta(n+1,o-n+2)$$

And

$$\chi_4 = \int_0^1 (1-l)\beta_l(n+1,o-n)dl$$

$$= \frac{1}{2}\beta(n+1,o-n) + \frac{1}{2}\beta(n+3,o-n) - \beta(n+3,o-n)$$

Hence the proof.

Theorem4

Let $\eta:[m_1,m_2] \subset (0,\infty) \to R$ is a differentiable function on (m_1,m_2) with the relation $m_1 < m_2$ such that $\eta' \in L(m_1,m_2)$ and $o > 0$ .if $|\eta'|^q$, where $q,l > 1$ with the relation $\frac{1}{q} + \frac{1}{l} = 1$, is a t-convex function then

$$\left|{}_1\Delta_n(m_1,m_2;o;\beta;J)\right| \le \frac{m_2^t - m_1^t}{2tm_2^t} w^{1/e}\left(w_1\left|\eta'(m_1)^q\right| + w_2\left|\eta'(m_2)\right|\right)^{1/q} \qquad (21)$$

Here

$$w = 2\int_0^{1/2}\left(\int_a^{1-a} s^n(1-s)^{o-n-1}ds\right)dl \quad ,$$

$$w_1 = \frac{m_2^{q(1-t)}}{2}\; {}_2F_1\left(q\left(1-\frac{1}{t}\right),2;3;1-\frac{m_1^t}{m_2^t}\right)$$

$$w_2 = \frac{m_2^{q(1-t)}}{2} \; {}_2F_1\left(q\left(1-\frac{1}{t}\right),1;3;1-\frac{m_1^t}{m_2^t}\right)$$

Proof

let $A_l = \left[lw_1^t + (1-l)m_2^t\right]$ here applying the Lemma1, Holder's inequality, and t-convexity of $\left|\eta'\right|^q$, we have

$$\begin{aligned}
\left|{}_1\Delta_n\left(m_1,m_2;o;\beta;J\right)\right| &= \left|\frac{m_2^t - m_1^t}{2tm_2^t}\right| \int_0^1 \left\{\beta_{1-l}\left(n+1,o-n\right) - \beta_l\left(n+1,o-n\right)\right\} \\
&\quad \times A_l^{\frac{1}{t}-1}\eta'\left(\left[lm_1^t + (1-l)m_2^t\right]^{1/t}\right)dl \\
&\le \frac{m_2^t - m_1^t}{2tm_2^t}\left(\int_0^1 \left|\beta_{1-l}\left(n+1,o-n\right) - \beta_l\left(n+1,o-n\right)\right|^{e\,1/e}\right) \\
&\quad \times \left(\int_0^1 A_l^{q\left(\frac{1}{t}-1\right)}\left|\eta'\left[lm_1^t + (1-l)m_2^t\right]^{1/t}\right|^q dl\right)^{1/q} \\
&\le \frac{m_2^t - m_1^t}{2tm_2^t} w^{1/t}\left(\int_0^1 A_l^{q\left(\frac{1}{t}-1\right)}\left[l\left|\eta'\left(m_1\right)^q\right| + (1-l)\left|\eta'\left(m_2\right)\right|^q\right]^{1/q} dl\right) \\
&= \frac{m_2^t - m_1^t}{2tm_2^t} w^{1/t}\left(w_1\left|\eta'\left(m_1\right)\right|^q + w\left|\eta'\left(m_2\right)\right|^q\right)^{1/q} \qquad (22)
\end{aligned}$$

Where

$$\begin{aligned}
w &= \int_0^1 \left|\beta_{1-l}\left(n+1,o-n\right) - \beta_l\left(n+1,o-n\right)\right|^e dl \\
&= \int_0^{1/2} \left|\beta_{1-l}\left(n+1,o-n\right) - \beta_l\left(n+1,o-n\right)\right|^e dl \\
&\quad + \int_{1/2}^1 \left|\beta_l\left(n+1,o-n\right) - \beta_{1-l}\left(n+1,o-n\right)\right|^e dl \\
&= \int_0^{1/2}\left(\int_l^{1-l} s^n(1-s)^{o-n-1}ds\right)^e dl + \int_{1/2}^1\left(\int_{1-l}^{l} s^n(1-s)^{o-n-1}ds\right)^e dl \\
&= 2\int_0^1\left(\int_l^{1-l} s^n(1-s)^{o-n-1}ds\right)^e dl
\end{aligned}$$

$$w_1 = \int_0^1 lA_l^{q\left(\frac{1}{t}-1\right)}dl = \frac{m_2^{q(1-t)}}{2} \; {}_2F_1\left(q\left(1-\frac{1}{t}\right),2;3;1-\frac{m_1^t}{m_2^t}\right)$$

Proof is completed.

III.INEQUALITY OF HERMITE-HADAMARD-FEJER

In this section of paper, we established Hermite-Hadamard Fejer inequalities for new f-divergence measure with the help of conformable fraction integral mapping.

Definition5 [14]

Let the $t \in R\setminus\{0\}$ .A function $\eta:[m_1,m_2] \subseteq (0,\infty) \to R$ is called t-symmetric around $\left[\frac{m_1^t + m_2^t}{2m_2^t}\right]$ if

$$\eta(a) = \eta'\left(\left[m_1^t + m_2^t - a^t\right]^{1/t}\right) \quad \forall a \in \left[m_1^t, m_2^t\right]$$

Using this result prove the following Lemma.

Lemma2

Let $t \in R \setminus \{0\}$. If $\eta : [m_1, m_2] \subseteq (0,\infty) \to R$ be a integrable and t-symmetric with $\left[\frac{m_1^t + m_2^t}{2m_2^t}\right]^{1/t}$, then

i.For $t > 0$, have this result.

$$J_o^{m_1^t}(\eta(\phi))(m_2^t) = J_o^{m_2^t}(\eta(\phi))(m_1^t)$$

$$= \frac{1}{2}\left[J_o^{m_1^t}(\eta(\phi))(m_2^t) + J_o^{m_2^t}(\eta(\phi))(m_1^t)\right] \quad (23)$$

With $o > 0$ and $\phi(s) = s^{1/t}$ $\forall s \in [m_2^t, m_1^t]$

ii. Lemma 2 for $t < 0$, then we have

$$J_o^{m_2^t}(\eta(\phi))(m_1^t) = J_o^{m_1^t}(\eta(\phi))(m_2^t)$$

$$= \frac{1}{2}\left[J_o^{m_2^t}(\eta(\phi))(m_1^t) + J_o^{m_1^t}(\eta(\phi))(m_2^t)\right] \quad (24)$$

Proof

Since η be a t-symmetric around $\left[\frac{m_1^t + m_2^t}{2m_2^t}\right]^{1/t}$, using the definition $\eta(a) = \eta\left(\left[m_1^t + m_2^t - a^t\right]^{1/t}\right)$ $\forall a \in [m_1^t, m_2^t]$ now set the variable $s = m_1^t + m_2^t - a$

Gives

$$J_o^{m_1^t}(\eta(\phi))(m_2^t) = \frac{1}{\lfloor n}\int_{m_1^t}^{m_2^t}(m_2^t - s)^n (s - m_1^t)^{t-n-1}\eta\left(s^{1/t}\right)ds$$

$$= \frac{1}{\lfloor n}\int_{m_1^t}^{m_2^t}(a - m_1^t)^n (m_2^t - a)^{o-n-1}\eta\left(\left[m_1^t + m_2^t - a\right]\right)da$$

$$= \frac{1}{\lfloor n}\int_{m_1^t}^{m_2^t}(a - m_1^t)^n (m_2^t - a)^{o-n-1}\eta\left(a^{1/t}\right)da = J_o^{m_2^t}(\eta(\phi))(m_1^t) \quad (25)$$

The proof is completed.

(ii) Proof is similar to (i).
Corollary 1
by the assumption of lemma 3
1. If $t = 1$ in (i) the result is

$$J_o^{m_1}\eta(m_2) = J_o^{m_2}\eta(m_1) = \frac{1}{2}\left[J_o^{m_1}\eta(m_2) + J_o^{m_2}\eta(m_1)\right] \quad (26)$$

2. If $t = -1$ in (ii) then we have

$$J_o^{m_1}\eta\left(1/m_2\right) = J_o^{m_2}\eta\left(1/m_1\right) = \frac{1}{2}\left[J_o^{m_1}\eta(1/m_2) + J_o^{m_2}\eta\left(1/m_1\right)\right] \quad (27)$$

Theorem4 [15]
Let $t \in R \setminus \{0\}$. If $\eta : [m_1, m_2] \subseteq (0,\infty) \to R$ be a t-convex function with the inequality $m_1 < m_2$ and $\eta \in [m_1, m_2]$ .if $\eta : [m_1, m_2] \subseteq R \setminus \{0\} \to R$ be a non-negative integrable and t-symmetric around the $\left[\frac{m_1^t + m_2^t}{2m_2^t}\right]^{1/t}$, then

a) For $t > 0$ have following inequality

$$\eta\left(\left[\frac{m_1^t + m_2^t}{2m_2^t}\right]^{1/t}\right)\left[J_o^{m_1^t}(\eta(\phi))(m_2^t) + J_o^{m_2^t}(\eta(\phi))(m_1^t)\right]$$

$$\leq\left[J_{o}^{m_1^t}\left(\eta(\phi)\right)\left(m_2^t\right)+J_{o}^{m_2^t}\left(\eta(\phi)\right)\left(m_1^t\right)\right]$$

$$\leq\frac{\eta(m_1)+\eta(m_2)}{2m_2^t}\left[J_{o}^{m_1^t}\left(\eta(\phi)\right)\left(m_2^t\right)+J_{o}^{m_2^t}\left(\eta(\phi)\right)\left(m_1^t\right)\right] \tag{28}$$

With $o>0$ and $\phi(a)=a^{1/t}\,\forall a\in\left[m_1^t,m_2^t\right]$

b) $t<0$, following inequalities

$$\eta\left(\left[\frac{m_1^t+m_2^t}{2m_2^t}\right]\right)\left[J_{o}^{m_2^t}\left(\eta(\phi)\right)\left(m_1^t\right)+J_{o}^{m_1^t}\left(\eta(\phi)\right)\left(m_2^t\right)\right]$$

$$\leq\left[J_{o}^{m_2^t}\left(\eta(\phi)\right)\left(m_1^t\right)+J_{o}^{m_1^t}\left(\eta(\phi)\right)\left(m_2^t\right)\right]$$

$$\leq\frac{\eta(m_1)+\eta(m_2)}{2m_2^t}\left[J_{o}^{m_2^t}\left(\eta(\phi)\right)\left(m_1^t\right)+J_{o}^{m_1^t}\left(\eta(\phi)\right)\left(m_2^t\right)\right] \tag{29}$$

Proof

η be a t-convex on $[m_1,m_2]$ then we have,

$$\eta\left(\left[\frac{a^t+b^t}{2}\right]^{1/t}\right)\leq\frac{\eta(a)+\eta(b)}{2}$$

Now take the values $a^t=lm_1^t+(1-l)m_2^t$ and $b^t=(1-l)m_1^t+lm_2^t$ for $l\in(0,1)$, then we get,

$$\eta\left(\left[\frac{m_1^t+m_2^t}{2m_2^t}\right]^{1/t}\right)\leq\frac{\eta\left[lm_1^t+(1-l)m_2^t\right]^{1/t}+\eta\left[(1-l)m_1^t+lm_2^t\right]^{1/t}}{2} \tag{30}$$

Multiplying this equation by $\frac{1}{\underline{\lfloor n}}l^n(1-l)^{o-n-1}\eta\left[lm_1^t+(1-l)m_2^t\right]^{1/t}$ on both sides, $o>0$ and then integrating about l over the $[0,1]$, we obtain

$$\frac{2}{\underline{\lfloor n}}\eta\left(\left[\frac{m_1^t+m_2^t}{2m_2^t}\right]^{1/t}\right)\int_0^1 l^n(1-l)^{o-n-1}\eta\left[lm_1^t+(1-l)m_2^t\right]dl$$

$$\leq\frac{1}{\underline{\lfloor n}}\int_0^1 l^n(1-l)^{o-n-1}\eta\left[lm_1^t+(1-l)m_2^t\right]^{1/t}\eta\left[lm_1^t+(1-l)m_2^t\right]^{1/t}dl$$

$$+\frac{1}{\underline{\lfloor n}}\int_0^1 l^n(1-l)^{o-n-1}\eta\left[(1-l)m_1^t+lm_2^t\right]^{1/t}\eta\left[lm_1^t+(1-l)m_2^t\right]^{1/t}dl \tag{31}$$

Since η is a non-negative integrable and t-symmetric with respect to $\left[\frac{m_1^t+m_2^t}{2m_2^t}\right]^{1/t}$, then.

$$\eta\left(\left[lm_1^t+(1-l)m_2^t\right]^{1/t}\right)=\eta\left[lm_2^t+(1-t)m_1^t\right]^{1/t}$$

Now choose the $s=lm_1^t+(1-l)m_2^t$, then,

$$\frac{2m_2^t}{\underline{\lfloor n}\left(m_2^t-m_1^t\right)^o}\eta\left[\frac{m_2^t+m_1^t}{2m_2^t}\right]^{1/t}\int_{m_1^t}^{m_2^t}\left(m_2^t-s\right)^n\left(s-m_1^t\right)^{o-n-1}ds$$

$$\leq\frac{1}{\underline{\lfloor n}\left(m_2^t-m_1^t\right)}\left[\int_{m_1^t}^{m_2^t}\left(m_2^t-s\right)^n\left(s-m_2^t\right)^{o-n-1}\eta\left(s^{1/t}\right)\eta\left(s^{1/t}\right)\right]ds$$

$$+\left[\int_{m_1^t}^{m_2^t}\left(m_2^t-s\right)^n\left(s-m_2^t\right)^{o-n-1}\eta\left(\left[m_1^t+m_2^t-s\right]^{1/t}\right)\eta\left(s^{1/t}\right)ds\right]$$

$$= \frac{1}{\lfloor n \left(m_2^t - m_1^t\right)^o} \left[\int_{m_1^t}^{m_2^t} \left(m_2^t - s\right)^n \left(s - m_1^t\right)^{o-n-1} \eta\left(s^{1/t}\right) \eta\left(s^{1/t}\right) ds \right.$$

$$\left. + \int_{m_1^t}^{m_2^t} \left(s - m_1^t\right)\left(m_2^t - s\right)^{o-n-1} \eta\left(s^{1/t}\right) \eta\left[m_1^t + m_2^t - s\right]^{1/t} ds \right] \quad (32)$$

by the lemma2 we have

$$\frac{m_2^t}{\left(m_1^t - m_2^t\right)} \eta\left(\left[\frac{m_1^t + m_2^t}{2m_2^t}\right]^{1/t}\right)\left[J_o^{m_1^t}\left(\eta(\phi)\right)\left(m_1^t\right) + J_o^{m_2^t}\left(\eta(\phi)\right)\left(m_2^t\right)\right]$$

$$\leq \frac{1}{\left(m_1^t - m_2^t\right)^o}\left[J_o^{m_1^t}\left(\eta(\phi)\right)\left(m_2^t\right) + J_o^{m_2^t}\left(\eta(\phi)\right)\left(m_1^t\right)\right] \quad (33)$$

This result completes the inequality (28) and for the second inequality, if η is a t-convex function then we have,

$$\eta\left(\left[lm_1^t + (1-l)m_2^t\right]^{1/t}\right) + \eta\left(\left[lm_2^t + (1-l)m_1^t\right]^{1/t}\right) \leq \left[\eta(m_1) + \eta(m_2)\right] \quad (34)$$

Multiplying by $\frac{1}{\lfloor n} l^n (1-l)^{o-n-1}\left(\left[lm_1^t + (1-l)m_2^t\right]^{1/t}\right)$ on both side of the equation (34) and integrating about l over $[0,1]$ then we have

$$\frac{1}{\lfloor n}\int_0^1 l^n (1-l)^{o-n-1} \eta\left(\left[lm_1^t + (1-l)m_2^t\right]^{1/t}\right)\eta\left(\left[lm_1^t + (1-l)m_2^t\right]\right) dl$$

$$+ \frac{1}{\lfloor n}\int_0^1 l^n (1-l)^{o-n-1} \eta\left(\left[lm_2^t + (1-l)m_1^t\right]^{1/t} \eta\left[lm_1^t + (1-l)m_2^t\right]^{1/t} dl\right)$$

$$\leq \left[\eta(m_1) + \eta(m_2)\right] \frac{1}{\lfloor n}\int_0^1 l^n (1-l)^{0-n-1} \eta\left[lm_1^t + (1-l)m_2^t\right] dl \quad (35)$$

That is

$$\frac{1}{\left(m_2^t - m_1^t\right)}\left[J_o^{m_1^t}\eta\left(\phi\left(m_2^t\right)\right) + J_o^{m_2^t}\eta\left(\phi\left(m_1^t\right)\right)\right]$$

$$\leq \frac{1}{\left(m_2^t - m_1^t\right)}\left[\frac{\eta(m_1) + \eta(m_2)}{2}\right]\left[J_o^{m_1^t}\eta\left(\phi\left(m_2^t\right)\right) + J_o^{m_2^t}\eta\left(\phi\left(m_1^t\right)\right)\right] \quad (36)$$

The proof is complete.
(ii) proof is similar to of (i).

Corollary2
Under the assumption of theorem4

1. If $t = 1$ then

$$\eta\left(\frac{m_1 + m_2}{2}\right)\left[J_o^{m_1}\eta(m_2) + J_o^{m_2}\eta(m_1)\right] \leq \left[J_o^{m_1}\eta\left(\eta'(m_2)\right) + J_o^{m_2}\eta\eta'\left((m_1)\right)\right]$$

$$\leq \frac{\eta(m_1) + \eta(m_2)}{2}\left[J_o^{m_1}\eta(m_2) + J_o^{m_2}\eta(m_1)\right] \quad (37)$$

2. If $t = -1$ then

$$\eta\left(\frac{2m_1m_2}{m_1 + m_2}\right)\left[J_o^{1/m_1}\eta\left(\phi\left(1/m_1\right)\right) + J_o^{1/m_2}\eta\left(\phi\left(1/m_1\right)\right)\right]$$

$$\leq \left[J_o^{1/m_1}\eta\left(\phi\left(1/m_1\right)\right) + J_o^{1/m_2}\eta\left(\phi\left(1/m_1\right)\right)\right]$$

$$\leq \frac{\eta(m_1) + \eta(m)}{2}\left[J_o^{1/m_1}\eta\left(\phi\left(1/m_1\right)\right) + J_o^{1/m_2}\eta\left(\phi\left(1/m_1\right)\right)\right] \quad (38)$$

Lemma3

Let $t \in R \setminus \{0\}$ If $\eta : [m_1, m_2] \subseteq (0,\infty) \to R$ be a differential function and $\eta \in [m_1, m_2]$ if $\eta : [m_1, m_2] \subseteq R \setminus \{0\} \to R$ be a non-negative integrable and t-symmetric around the $\left[\frac{m_1^t + m_2^t}{2m_2^t}\right]^{1/t}$ then

a) $t > 0$, following inequalities

$$\frac{\eta(m_1)+\eta(m_2)}{2}\left[J_o^{m_1^t}\left(\eta(\phi)(m_2^t)\right)+J_o^{m_2^t}\left(\eta(\phi)(m_1^t)\right)\right]$$
$$-\left[J_o^{m_1^t}\eta\left(\eta'(\phi)(m_2^t)\right)+J_o^{m_2^t}\eta\left(\eta'(\phi)(m_1^t)\right)\right]$$
$$\leq \frac{1}{\lfloor n}\int_{m_1^t}^{m_2^t}\left[\int_{m_1^t}^{x}(m_2^t-y)^n(y-m_1^t)^{o-n-1}\eta(\phi(y))dy-\int_{x}^{m_2^t}(y-m_1^t)^n(m_2^t-y)^{o-n-1}\eta(\phi(y))dy\right]$$
$$\eta'(\phi(x))dx \tag{39}$$

With $o > 0$ and $\varphi(a) = a^{1/t}\, \forall a \in [m_1^t, m_2^t]$

b) $t < 0$, following inequalities

$$\frac{\eta(m_1)+\eta(m_2)}{2}\left[J_o^{m_2^t}\left(\eta(\phi)(m_1^t)\right)+J_o^{m_1^t}\left(\eta(\phi)(m_1^t)\right)\right]$$
$$-\left[J_o^{m_2^t}\eta\left(\eta'(\phi)(m_1^t)\right)+J_o^{m_1^t}\eta\left(\eta'(\phi)(m_2^t)\right)\right]$$
$$\leq \frac{1}{\lfloor n}\int_{m_2^t}^{m_1^t}\left[\int_{m_2^t}^{x}(m_1^t-y)^n(y-m_1^t)^{o-n-1}\eta(\phi(y))dy-\int_{x}^{m_1^t}(y-m_1^t)^n(m_2^t-y)^{o-n-1}\eta(\phi(y))dy\right]$$
$$\eta'(\phi(x))dx \tag{40}$$

Where $\varphi(a) = a^{1/t}\, \forall a \in [m_2^t, m_1^t]$

Proof

a)
$$I' = \int_{m_1^t}^{m_2^t}\left(\int_{m_1^t}^{x}(m_2^t-y)^n(y-m_1^t)^{o-n-1}\eta(\phi(y))dy\right)\eta(\phi(x))dx$$
$$-\int_{m_1^t}^{m_2^t}\left(\int_{y}^{m_2^t}(y-m_1^t)^n(m_2^t-y)^{o-n-1}\eta(\phi(y))dy\right)\eta(\phi(x))dx$$
$$= I_1' - I_2' \tag{41}$$

Integrating by part and using the result of Lemma3 we get

$$I_1' = \left(\int_{m_1^t}^{x}(m_2^t-y)^n(y-m_1^t)^{o-n-1}\eta(\phi(y))dy\right)\eta(\phi(x))\Bigg|_{m_1^t}^{m_2^t}$$

$$I_1' = \left(\int_{m_1^t}^{x}(m_2^t-y)^n(y-m_1^t)^{o-n-1}\eta(\phi(y))dy\right)\eta(\phi(x))\Bigg|_{m_1^t}^{m_2^t} = \lfloor n\left[\eta(\phi(m_2^t))J_o^{m_1^t}\eta(\phi(m_2^t)) - J_o^{m_1^t}\left(\eta'\eta(\phi(m_2^t))\right)\right]$$

$$= \lfloor n\left[\frac{\eta(\phi(m_2^t))}{2}\left\{\left[J_o^{m_2^t}\eta(\phi(m_1^t))+J_o^{m_1^t}\eta(\phi(m_2^t))-\right]J_o^{m_1^t}\left(\eta'\eta(\phi(m_2^t))\right)\right\}\right] \tag{42}$$

Similarly

$$I_2' = \left(\int_{x}^{m_2^t}(y-m_1^t)^n(m_2^t-y)^{o-n-1}\eta(\phi(y))dy\right)\eta(\phi(x))\Bigg|_{m_1^t}^{m_2^t}$$

$$+\int_{m_1^t}^{m_2^t} \left(x-m_1^t\right)^n \left(m_2^t-x\right)^{o-n-1} \eta\left(\phi(x)\right)\eta^{\cdot}\left(\phi(x)\right)dx$$

$$=\lfloor n\left[-\eta\left(\phi\left(m_1^t\right)\right)J_o^{m_2^t}\eta\left(\phi\left(m_1^t\right)\right)-J_o^{m_1^t}\left(\eta^{\cdot}\eta\left(\phi\left(m_1^t\right)\right)\right)\right]$$

$$=\lfloor n\left[-\frac{\eta\left(\phi\left(m_1^t\right)\right)}{2}\left\{J_o^{m_2^t}\eta\left(\phi\left(m_1^t\right)\right)+J_o^{m_1^t}\eta\left(\phi\left(m_2^t\right)\right)+J_o^{m_2^t}\left(\eta^{\cdot}\eta\left(\phi\left(m_1^t\right)\right)\right)\right\}\right] \quad (43)$$

By the equation (42) and (43)

$$I^{\cdot}=I_1^{\cdot}-I_2^{\cdot}$$

$$=\lfloor n\left[\frac{\eta(m_1)+\eta(m_2)}{2}\left\{J_o^{m_1^t}\eta\left(\phi\left(m_2^t\right)\right)+J_o^{m_2^t}\eta\left(\phi\left(m_1^t\right)\right)\right\}\right]$$

$$-\left[J_o^{m_1^t}\eta^{\cdot}\left(\eta\left(\varphi\left(m_2^t\right)\right)\right)+J_o^{m_2^t}\eta^{\cdot}\left(\eta\left(\varphi\left(m_1^t\right)\right)\right)\right] \quad (44)$$

Multiplying the equation (44) by $\frac{1}{\lfloor n}$, then we get the equation (40)

(ii) proof is similar to that (i).

CONCLUSION

In this research paper we present the basic facts of HH and HH-f inequality for convex function and established some contribution of inequality theory and probability theory from the perspective of applications.

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