Hermite–Hadamard and Hermite–Hadamard Fejer inequality for new f-divergence measure via conformable fractional integrals

Abstract: In information geometry, a divergence measure is a type of statistical distance and a binary function which established the separation from one probability distribution to another on a statistical numerous. The basic use of divergence measure is statistical data processing, information storage, decision making etc. The most famous inequality regarding the integral mean of a convex function is Hermite-Hadamard's inequality and the weighted version of this is called the Hermite-Hadamard-Fejér inequality. Purpose of this paper is to find Hermite-Hadamard and Hermite-Hadamard Fejer type inequalities for new f-divergence measure with the help of conformable fractional integrals. Hermite-Hadamard inequality gives us necessary and sufficient condition for a function must be convex. Here we consider the new f-divergence measure, it has the property of convexity. In this research article we drive some inequalities for t-convex function which gives us the extensions of the previous work for convex and t-convex function and also obtains some fractional midpoint type inequalities. The main purpose of this paper is to establish conformal fractional approximation of Hermite-Hadamard and Hermite-Hadamard Fejer type inequalities for new f-divergence measure which close the fractional integral and the Riemann-Liouville integrable into single form, also gives us some new results for ψ-f-Riemann-Liouville integral as special cases of main results. This article gives us most useful link between convexity and symmetry.

Keywords: Hermite-Hadamard inequality, Hermite-Hadamard Fejer type inequality, Riemann-Liouville fractional integral, new f-divergence measure, t-convex function, conformable fraction integral.

I. INTRODUCTION

In the literature of information theory inequalities play a useful role in finding the relations between divergence measures, bounds, coding, and various field. Various mathematicians have used various types of inequalities. The Hermite-Hadamard inequality is one of the most significant inequality. Initially it was discovered by Hermite and afterward by Hadamard. Here, giving some examples of these type of inequalities:

Fractional Hermite-Hadamard-Fejer inequalities for a convex function with respect to an increasing function [1].


In the present article we will present some HH and HH-F inequalities for new f-divergence measure with the support of conformable fractional integral.

A function η defined $M \rightarrow \mathbb{R}$ and $m_1, m_2 \in M, l \in [0,1]$ is known as convex if the relation holds.

$$\eta\left(lm_1 + (1-l)m_2\right) \leq l\eta(m_1) + (1-l)\eta(m_2)$$

(1)

Many authors have given the importance of convex functions and their generalizations of the convex functions. The Hermite-Hadamard inequality defined in the interval.

$$\eta\left(\frac{m_1 + m_2}{2}\right) \leq \frac{1}{m_2 - m_1}\int_{m_1}^{m_2} \eta(s)ds \leq \frac{\eta(m_1) + \eta(m_2)}{2}$$

(2)

Here the function η is nonnegative symmetric and integrable also to $\frac{m_1 + m_2}{2}$, known as Hermite-Hadamard-Fejer inequality. Further generalization of the inequality (2) and (3) in the different ways not only classical integral $\forall m_1, m_2 \in M$ With $m_1 < m_2$. Then the following inequality proved by Fejer. [8].
\[ \eta\left(\frac{m_1 + m_2}{2}\right) \int_{m_1}^{m_2} \eta(s) ds \leq \frac{1}{m_2 - m_1} \int_{m_1}^{m_2} \eta(s) \eta(s) ds \leq \frac{\eta(m_1) + \eta(m_2)}{2} \int_{m_1}^{m_2} \eta(s) ds \]  

(3)

but also other generalized as Riemann-Liouville integrable, \( \psi \)-Riemann-Liouville and conformable fractional integral etc.

Definition 1 [8, 9]

Let the interval \( M \subset (0, \infty) \) and \( t \in R \setminus \{0\} \). Then the function \( \eta: M \rightarrow R \) called t-convex if

\[ \eta\left[\left[m_1^l + (1-t) m_2^r\right]^\alpha\right] \leq t \eta(m_1) + (1-t) \eta(m_2) \]

\[ \forall m_1, m_2 \in M \quad \text{And} \quad l \in [0,1] \]

Definition 2 [10]

Let \( \eta \in L(m_1, m_2) \) the left and right sided in the Reimann-Liouville fraction integrable \( J_{m_1}^m \eta \) and \( J_{m_2}^m \eta \) of order \( O \in C \) with \( R(o) > 0 \) and \( m_1 > m_2 \geq 0 \) are given by

\[ J_{m_1}^m \eta(s) = \frac{1}{\Gamma(a)} \int_{m_1}^{m} (s-w)^{a-1} \eta(w) dw, \quad s > m_1 \]

And

\[ J_{m_2}^m \eta(s) = \frac{1}{\Gamma(a)} \int_{m_2}^{m} (w-s)^{a-1} \eta(w) dw, \quad s < m_2 \]

Respectively, where \( \Gamma(.) \) be known as gamma function. The conformal fractional integral as follows.

Definition 3 [11]

Let \( o \in (n,n+1) \) and \( \gamma = o - n \), then the left and right sided conformable fractional integrals of order \( O > 0 \) are given by

\[ J_{m_1}^m \eta(s) = \frac{1}{\Gamma(n)} \int_{m_1}^{m} (s-w)^{n-1} \eta(w) dw \]

And

\[ J_{m_2}^m \eta(s) = \frac{1}{\Gamma(n)} \int_{m_2}^{m} (w-s)^{n-1} \eta(w) dw \]

Respectively

\[ \Rightarrow \quad \text{For} \quad o = n + 1 \quad \text{then} \quad \eta = 1 \]

Where \( n = 0,1,2,\ldots \) and in this case conformable fraction integrals becomes Riemann-Liouville fractional integrals. The classical beta function and hypergeometric function are defined, respectively, by

\[ \eta(m_1, m_2) = \int_{0}^{1} w^{m_1 - 1} (1-w)^{m_2 - 1} ds \]

and

\[ {}_2F_1(m_1, m_2; s, w) = \frac{1}{\beta(m_1, m_2; s-m_2)} \int_{0}^{s} (1-s)^{m_2-1} (1-sw)^{m_1-1} ds \]

With \( s > m_2 > 0, |w| < 1 \).

This is known as incomplete beta function.

\[ \beta(m_1, m_2) = \int_{0}^{1} w^{m_1-1} (1-w)^{m_2-1} dw, \quad w \in [0,1] \]

The classical beta function and the incomplete beta function is given as follows.

\[ \beta(m_1, m_2) = \beta(m_1, m_2) + \beta_{-\alpha}(m_1, m_2) \]

Definition 4 [12]

New f-divergence measure and its properties is given by

\[ S_f(P, Q) = \sum_{i=1}^{n} q_i f\left( \frac{p_i + q_i}{2q_i} \right) \]
A function \( f \) defined as \([0, \infty) \rightarrow R\) is the convex, then we have the following inequality 
\[ S_f(P, Q) \geq f(1) \]

II. HERMITE-HADAMARD INEQUALITIES

In this section, Established Hermite-Hadamard inequalities for new f-divergence measure with the help of conformable fraction integral mapping.

Theorem1 [13]
Let \( \eta : [m_1, m_2] \subset (0, \infty) \rightarrow R \) (here \( \eta = s, (p, q) \) is a t-convex function such that \( \eta \in L[m_1, m_2] \) and \( \theta > 0 \). Then

i. For \( t > 0 \), we have
\[ m_2 \eta \left[ \left( \frac{m_1^t + m_2^t}{2m_2^t} \right)^\frac{1}{t} \right] \leq \frac{t + 1}{2t} \left[ J^{m_1}(\eta(\phi))m_1^t + J^{m_2}(\eta(\phi)m_2^t) \right] \] 

Here \( \phi(s) = s^\theta \forall s \in [m_1^t, m_2^t] \)

ii. For \( t < 0 \), we have
\[ m_2 \eta \left[ \left( \frac{m_1^t + m_2^t}{2m_2^t} \right)^\frac{1}{t} \right] \leq \frac{t + 1}{2t} \left[ J^{m_1}(\eta(\phi))m_1^t + J^{m_2}(\eta(\phi)m_2^t) \right] \]

Here \( \phi(s) = s^\theta \forall s \in [m_2^t, m_1^t] \)

Proof

\( \eta \) be a t-convex function on \([m_1, m_2] \), we have

\[ b(\frac{a + b}{2}) \leq \frac{1}{2} \eta(a) + \frac{1}{2} \eta(b) \]

Now take \( a' = lm_1^t + (1-l)m_2^t \) and \( b' = (1-l)m_1^t + lm_2^t \) with \( l \in [0,1] \) then we get

\[ m_2 \eta \left[ \left( \frac{m_1^t + m_2^t}{2m_2^t} \right)^\frac{1}{t} \right] \leq \frac{1}{2} \left[ (lm_1^t + (1-l)m_2^t)\eta \left( \left( \frac{m_1^t + m_2^t}{2m_2^t} \right)^\frac{1}{t} \right) \right] \]

(7)

Multiplying with \( \frac{1}{n} l^t (1-l)^{n-t-1} \) equation (7), Here \( l \in (0,1), \theta > 0 \), on both sides and integrating about \( k \) over \([0,1] \) then we have

\[ \frac{2m_2}{n} \eta \left[ \left( \frac{m_1^t + m_2^t}{2m_2^t} \right)^\frac{1}{t} \right] \int_0^1 l^t (1-l)^{n-t-1} dl \leq \frac{1}{n} \int_0^1 l^t (1-l)^{n-t-1} \eta \left( \left( \frac{m_1^t + (1-l)m_2^t}{2m_2^t} \right)^\frac{1}{t} \right) dl \]

\[ + \frac{1}{n} \int_0^1 l^t (1-l)^{n-t-1} \eta \left( \left(1-l)m_1^t + lm_2^t \right)^\frac{1}{t} \right) = I_1 + I_2 \]

(8)

By putting the value \( s = lm_1^t + (1-l)m_2^t \) we have

\[ I_1 = \frac{1}{n} \int_0^1 l^t (1-l)^{n-t-1} \eta \left( \left( \frac{lm_1^t + (1-l)m_2^t}{2m_2^t} \right)^\frac{1}{t} \right) dl \]

\[ = \frac{1}{n} \int s \left( \frac{m_1^t - m_2^t}{m_1^t} \right)^{n-t-1} \left( \frac{s - m_2^t}{m_1^t - m_2^t} \right)^{n-t-1} \eta(\phi) \frac{ds}{m_1^t - m_2^t} \]

\[ = \frac{1}{n} \int_0^1 \left( \frac{m_2^t - s}{m_1^t} \right)^n \left( \frac{s - m_2^t}{m_1^t - m_2^t} \right)^{n-t-1} \eta(\phi) ds \]
\[ J_n^s \left( \eta(\phi) \right) \left( m_2' \right) \]

By putting \( s = \ln l' + (1 - l)m'_1 \), we have

\[ I_1 = \frac{1}{n} \int_0^s (1 - l')^{n-1} \eta \left[ \ln m'_1 + (1 - l)m'_1 \right]^V dl \]

\[ = \frac{1}{n} \int_0^s \left( s - m'_1 \right) \left( 1 - \frac{m'_1 - m'}{m'_1 - m'_i} \right)^{n-1} \left( \eta(\phi(s)) \right) \frac{ds}{m'_1 - m'_i} \]

\[ = \frac{1}{n} \int_0^s (s - m'_1) \left( m'_1 - u \right)^{n-1} \left( \eta(\phi(s)) \right) ds \]

\[ = \frac{1}{n} \left( m'_1 - m'_i \right)^{m'} J_s \left( \eta(\phi(s)) \right) \left( m'_i \right) \]

Now putting the value of \( I_1 \) and \( I_2 \) in the equation (9), the first inequality of (5) is proved, for other inequality, we note that,

\[ \eta \left[ \ln m'_1 + (1 - l)m'_2 \right]^V \eta \left[ \ln m'_1 + (1 - l)m'_2 \right]^V \leq \eta \left( m_1 \right) + \eta \left( m_2 \right) \]

Multiplying with \( \frac{1}{n} l' (1 - l)^{n-1} \), here \( l \in (0, 1) \), \( o > 0 \), on both sides and then integrating about \( l \) over \([0,1] \), hence the inequality (11). This completes the proof.

**Lemma:**

Suppose \( \eta : [m_1, m_2] \subset (0, \infty) \rightarrow R \) be a differentiable function on \( (m_1, m_2) \) with \( m_1 < m_2 \) such that,

I. For \( t > 0 \), we have

\[ \Delta_n (m_1, m_2; \beta, J) = m_2 \left( \frac{m'_1 - m'_i}{2m'_1} \right) \int_0^t \beta(t) \left( n + 1, o - n \right) - \beta(t) \left( n + 1, o - n \right) \times A^V_\eta \left[ \ln m'_1 + (1 - l)m'_2 \right]^V \]

II. For \( t < 0 \), we have

\[ \Delta_n (m_1, m_2; \beta, J) = m_2 \left( \frac{m'_1 - m'_i}{2m'_1} \right) \int_0^t \beta(t) \left( n + 1, o - n \right) - \beta(t) \left( n + 1, o - n \right) \times B^V_\eta \left[ \ln m'_1 + (1 - l)m'_2 \right]^V \]

Proof

\[ \int_0^t \beta(t) \left( n + 1, o - n \right) - \beta(t) \left( n + 1, o - n \right) A^V_\eta \left[ \ln m'_1 + (1 - l)m'_2 \right]^V dl \]

\[ = \int_0^t \beta(t) \left( n + 1, o - n \right) A^V_\eta \left[ \ln m'_1 + (1 - l)m'_2 \right]^V dl - \int_0^t \beta(t) \left( n + 1, o - n \right) A^V_\eta \left[ \ln m'_1 + (1 - l)m'_2 \right]^V dl \]

\[ = I_1 - I_2 \]
Using by part integration then we have

\[ I_1 = \int_0^1 \beta \left( n+1, o-n \right) \left \{ m_1 \eta \left[ \left( m_1 + (1-l) m_2 \right)^\gamma \right] - \left( m_2 \right)^\gamma \right \} dl \]

\[ = \int_0^1 \left( \int_0^1 s^\gamma \left( 1-s \right)^{\gamma-1} ds \right) \left \{ m_1 \eta \left[ \left( m_1 + (1-l) m_2 \right)^\gamma \right] - \left( m_2 \right)^\gamma \right \} dl \]

\[ = \frac{tm_1'}{m_2'-m_1'} \beta \left( n+1, o-n \right) \eta \left( m_1 \right) \left( \begin{vmatrix} 1 - a - m_2/m_1' \\ m_1' - m_2' \end{vmatrix} \right)^{\gamma-1} \eta \left( \phi \left( x \right) \right) dx \]

\[ = \frac{tm_1'}{m_2'-m_1'} \beta \left( n+1, o-n \right) \eta \left( m_1 \right) \left( \begin{vmatrix} 1 - a - m_2/m_1' \\ m_1' - m_2' \end{vmatrix} \right)^{\gamma-1} J_{n+1}^{\gamma} \eta \left( \phi \left( x \right) \right) \]

Similarly, we have

\[ I_2 = \int_0^1 \beta \left( n+1, o-n \right) \eta \left[ \left( m_1 + (1-l) m_2 \right)^\gamma \right] dl \]

\[ = \int_0^1 \left( \int_0^1 s^\gamma \left( 1-s \right)^{\gamma-1} ds \right) \eta \left[ \left( m_1 + (1-l) m_2 \right)^\gamma \right] dl \]

\[ = \frac{tm_1'}{m_2'-m_1'} \beta \left( n+1, o-n \right) \eta \left( m_1 \right) + \frac{tm_2'}{m_2'-m_1'} \beta \left( n+1, o-n \right) \eta \left( m_2 \right) \left( \begin{vmatrix} 1 - a - m_2/m_1' \\ m_1' - m_2' \end{vmatrix} \right)^{\gamma-1} \eta \left( \phi \left( x \right) \right) dx \]

Second proof is similar to first.

Theorem 2
Let \( \eta : [m_1, m_2] \subset (0, \infty) \rightarrow R \) be a differentiable function on \( [m_1, m_2] \) with \( m_1 < m_2 \) such that \( \eta \in L[1, 2, 1, 2] \) and \( \alpha > 0 \) if \( \left[ \gamma \right] \), where \( q \geq 1 \) is a convex function, then

i. For \( t > 0 \), we have

\[ |\Delta \left( m_1, m_2; \alpha, \beta; J \right)| \leq \frac{m_1' - m_2'}{2 m_2'} \chi_{J} \left( \eta \left( m_1 \right) \right)^\gamma + \chi_{J} \left( \eta \left( m_2 \right) \right)^\gamma \]

\[ \chi_3 = \frac{m_1'}{2} F_1 \left( \frac{1}{3}; 1, 2, 3; 1, \frac{m_1'}{m_2'} \right) \]

\[ \chi_4 = \frac{m_1'}{2} F_1 \left( \frac{1}{3}; 1, 2, 3; 1, \frac{m_2'}{m_1'} \right) \]

ii. For \( t < 0 \), we have

\[ |\Delta \left( m_1, m_2; \alpha, \beta; J \right)| \leq \frac{m_2' - m_1'}{2 m_1'} \chi_{J} \left( \eta \left( m_1 \right) \right)^\gamma + \chi_{J} \left( \eta \left( m_2 \right) \right)^\gamma \]

\[ \chi_5 = \beta \left( n+1, o-n+1 \right) - \beta \left( n+2, o-n \right) \]

\[ \chi_6 = \frac{m_1'}{2} F_1 \left( \frac{1}{3}; 1, 2, 3; 1, \frac{m_1'}{m_2'} \right) \]

\[ \chi_7 = \frac{m_2'}{2} F_1 \left( \frac{1}{3}; 1, 2, 3; 1, \frac{m_2'}{m_1'} \right) \]
\[ X_5 = \frac{m_i^{l-1}}{2} \, \text{F} \left( 1 - \frac{1}{2}, 2; 3; 1 - \frac{m_i}{m_j} \right) \]

Proof
i. \( A_k = \left[ \text{ln} + (1-t) m_k^l \right] \)

Applying the Lemma 1, power mean inequality and t-convexity of \( |\eta| \), we find

\[ \left| A_n \left( m_i, m_j; \alpha; \beta; J \right) \right| \leq \frac{m_j^l - m_i^l}{2 m_j^l} \, \left[ \beta_n \left( (n+1, o-n) - \beta_n (n+1, o-n) \right) + A_k \left( m_j^l \right) \right] \]

\[ \times \left[ \int \left[ \beta_n \left( (n+1, o-n) - \beta_n (n+1, o-n) \right) \right] \right]^\frac{1}{q} \, dt \]

\[ \leq \frac{m_j^l - m_i^l}{2 m_j^l} \, \left[ X_1 \left( m_j^l \right) + X_2 \left( m_j^l \right) \right] \]

(19)

Where

\[ X = \int \left( \beta_n \left( (n+1, o-n) - \beta_n (n+1, o-n) \right) \right) \, dt \]

\[ = \int \left[ \int s^n \left( \left( 1-s \right)^{n-1} - 1 \right) \right] \, ds + \int \left[ \int s^n \left( (1-s)^{n-1} - 1 \right) \right] \, dt \]

\[ = s \left[ \int s^n \left( 1-s \right)^{n-1} \right] + \int \left( 1-s \right)^{n-1} \, dt \]

\[ + \int \left[ \int s^n \left( 1-s \right)^{n-1} \right] \, ds + \int \left( 1-s \right)^{n-1} \, dt \]

\[ \beta_n \left( (n+1, o-n+1) - \beta_n (n+1, o-n) + \beta (n+2, o-n) \right) \]

\[ X_1 = \frac{1}{2} A_k \left( m_j^l \right) \]

\[ = \frac{m_j^l - m_i^l}{2 m_j^l} \, \text{F} \left( 1 - \frac{1}{2}, 2; 3; 3 - \frac{m_i}{m_j} \right) \]

And

\[ X_2 = \frac{1}{2} (1-t) A_k \left( m_j^l \right) \]

\[ = \frac{m_j^l - m_i^l}{2 m_j^l} \, \text{F} \left( 1 - \frac{1}{2}, 1; 3; 1 - \frac{m_i}{m_j} \right) \]

Hence the proof

Proof (ii) is similar to (i)

Theorem 3
Let \( \eta : \left[m_i, m_j\right] \subset (0, \infty) \rightarrow R \) is a differentiable function on \( (m_i, m_j) \) with the relation \( m_i < m_j \) such that

\[ \eta \in L (m_i, m_j) \] and \( O > 0 \).

If \( |\eta|^{\rho} \) where \( q > 1 \) is a t-convex function then for \( t > 0 \), we have

\[ \left| A_2 \left( m_i, m_j; \alpha; \beta; J \right) \right| \leq \frac{m_j^l - m_i^l}{m_j^l} \, \rho \left( \left( \rho_i - \rho_j \right)|\eta (m_i)^{\rho} + (\rho_j - \rho_j)|\eta (m_j)^{\rho} \right] \]

Here

\[ X = \frac{m_j^l - m_i^l}{2 m_j^l} \, \text{F} \left( 1 - \frac{1}{2}, 1; 3; 1 - \frac{m_i}{m_j} \right) \]

\[ X_1 = \frac{1}{2} \beta (n+1, o-n+2) \]
\[ \chi_2 = \frac{1}{2} \beta(n+1,o-n) - \beta(n+3,o-n) \]
\[ \chi_3 = \frac{1}{2} \beta(n+2,o-n+1) - \frac{1}{2} \beta(n+1,o-n+2) \]

And
\[ \chi_4 = \frac{1}{2} \beta(n+1,o-n) + \frac{1}{2} \beta(n+3,o-n) - \beta(n+2,o-n) \]

Proof

Applying the result of Lemma 1, power mean inequality and t-convex of \( \| \eta \| \) we have

\[ |\Delta_2(m_1,m_2,o: \beta; J)| = \left| \frac{m_1 - m_2}{2m_2} \int_0^1 \left( \beta_{-t}(n+1,o-n) - \beta_t(n+1,o-n) \right) \times \frac{\chi^{1/2}}{\eta(m_1')} \left( \eta(m_1) + (1-t)\eta(m_2) \right)^{1/2} \right| \]

\[ \leq \frac{m_1 - m_2}{2m_2} \left( \chi \int_0^1 \left| \beta_{-t}(n+1,o-n) - \beta_t(n+1,o-n) \right| \times \left[ \eta(m_1') \right]^{1/2} \left( \eta(m_1) + (1-t)\eta(m_2) \right)^{1/2} \right) \]

\[ \leq \frac{m_1 - m_2}{2m_2} \chi \left( \int_0^1 \left( \chi - \chi_{x2} \right) \left| \eta(m_1') \right| + (\chi_{x3} - \chi) \left| \eta(m_2') \right| \right)^{1/2} \]

(20)

Where

\[ \chi = \left( \chi_{x1} \right)^{1/2} = \frac{m_1^{1/2}}{2} \left( \frac{1}{t} \right) \left( 1, 2 ; 1 ; \frac{m_2}{m_1} \right) \]

\[ \chi_1 = \left( \left( \beta_{-t}(n+1,o-n) \right) \right) \int_0^1 \frac{1}{l} \beta_t(n+1,o-n) = \frac{1}{2} \beta(n+1,o-n) \]

\[ \chi_2 = \frac{1}{2} \beta(n+1,o-n+2) \]

\[ \chi_3 = \beta(n+2,o-n+1) - \frac{1}{2} \beta(n+1,o-n+2) \]

\[ \chi_4 = \frac{1}{2} \beta(n+1,o-n) + \frac{1}{2} \beta(n+3,o-n) - \beta(n+2,o-n) \]

Hence the proof.

Theorem 4

Let \( \eta : [m_1, m_2] \subset (0, \infty) \rightarrow R \) be a differentiable function on \( (m_1, m_2) \) with the relation \( m_1 < m_2 \) such that

\[ \eta \in L(m_1, m_2) \quad \text{and} \quad \alpha > 0 \quad \text{if} \quad \frac{1}{q} + \frac{1}{l} = 1 \]

is a t-convex function then

\[ |\Delta_2(m_1,m_2,o: \beta; J)| \leq \frac{m_1 - m_2}{2m_2} \left( w \left( \left| \eta(m_1) \right|^q \right) + w_2 \left( \left| \eta(m_2) \right|^q \right) \right) \]

(21)

Here

\[ w = 2 \int_{k}^{1/2} \left( \int_0^1 s^{1-s} (1-s)^{q-1} \right) ds \]

\[ w_1 = \frac{m_2^{l-1}}{2} \left( q \left( 1 - \frac{1}{l} \right) + 2, 3 ; 1 - \frac{m_2}{m_1} \right) \]

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\[ w_2 = \frac{m_2^{(1-l)}}{2} \text{ } _2F_1\left(q\left(1 - \frac{1}{t}\right), 1; 3; 1 - \frac{m_1}{m_2}\right) \]

Proof

let \( A_1 = \left[l m'_1 + (1-l) m'_2\right] \) here applying the Lemma1, Holder's inequality, and t-convexity of \( |\eta|^\gamma \), we have

\[
|\Delta, (m_1, m_2; \alpha; \beta; J)| = \left|\frac{m'_1 - m'_2}{2m'_2}\right| \int_0^1 \left[|l \Delta_{t-1} (n+1, o-n) - \beta_t (n+1, o-n)|^\gamma\right] dl
\]

\[
\leq \frac{m'_1 - m'_2}{2m'_2} \left( \int_0^1 \left[l \Delta_{t-1} (n+1, o-n) - \beta_t (n+1, o-n)\right]^\gamma\right) \eta dl
\]

\[
\leq \frac{m'_1 - m'_2}{2m'_2} w^{\frac{\gamma}{m}} \left( \int_0^1 \left[l \Delta_{t-1} (n+1, o-n) - \beta_t (n+1, o-n)\right]^\gamma\right) \eta^\gamma dl
\]

\[
= \frac{m'_1 - m'_2}{2m'_2} w^{\frac{\gamma}{m}} \left( w_1 |\eta|^\gamma + w_2 |\eta (m_2)|^\gamma\right)^{\frac{\gamma}{m}}
\]

Where

\[
w = \int_0^1 |l \Delta_{t-1} (n+1, o-n) - \beta_t (n+1, o-n)|^\gamma\ dl
\]

\[
= \int_0^1 |l \Delta_{t-1} (n+1, o-n) - \beta_t (n+1, o-n)|^\gamma\ dl
\]

\[
+ \int_0^{\frac{1}{2}} |l \Delta_{t-1} (n+1, o-n) - \beta_t (n+1, o-n)|^\gamma\ dl
\]

\[
= \int_0^{\frac{1}{2}} \left( s^\alpha (1-s)^{\omega-n-1} \right) dl + \int_0^{\frac{1}{2}} \left( s^\omega (1-s)^{\omega-n-1} \right) dl
\]

\[
= 2 \int_0^{\frac{1}{2}} s^\omega (1-s)^{\omega-n-1} dl
\]

\[
w_1 = \int_0^1 \Delta_{t-1} (n+1, o-n) dl = \frac{m_2^{(1-l)}}{2} \text{ } _2F_1\left(q\left(1 - \frac{1}{t}\right), 1; 3; 1 - \frac{m_1}{m_2}\right)
\]

Proof is completed.

\[\text{III. INEQUALITY OF HERMITE-HADAMARD-FEJER}\]

In this section of paper, we established Hermite-Hadamard Fejer inequalities for new f-divergence measure with the help of conformable fraction integral mapping.

Definition 5 [14]

Let the \( t \in R \setminus \{0\} \) A function \( \eta : [m_1, m_2] \subseteq (0, \infty) \to R \) is called t-symmetric around \( \left[\frac{m'_1 + m'_2}{2m'_2}\right] \) if

\[ \eta(a) = \eta\left([m'_1 + m'_2 - a]^{\frac{1}{m}}\right) \quad \forall a \in [m'_1, m'_2] \]

Using this result prove the following Lemma.

Lemma 2
Let \( t \in R \setminus \{0\} \). If \( \eta : [m_1, m_2] \subseteq (0, \infty) \rightarrow R \) be a integrable and t-symmetric with \( \left[ \frac{m_1^t + m_2^t}{2m_2^t} \right]^{\frac{1}{t}} \), then

i. For \( t > 0 \), have this result.

\[
J^\eta_t(\eta(\phi))(m_2^t) = J^\eta_t(\eta(\phi))(m_1^t)
\]

\[
= \frac{1}{2} \left[ J^\eta_t(\eta(\phi))(m_2^t) + J^\eta_t(\eta(\phi))(m_1^t) \right]
\]

With \( o > 0 \) and \( \phi(s) = s^{\frac{1}{t}} \), \( \forall s \in [m_2^t, m_1^t] \)

ii. Lemma 2 for \( t < 0 \), then we have

\[
J^\eta_t(\eta(\phi))(m_2^t) = J^\eta_t(\eta(\phi))(m_1^t)
\]

\[
= \frac{1}{2} \left[ J^\eta_t(\eta(\phi))(m_2^t) + J^\eta_t(\eta(\phi))(m_1^t) \right]
\]

Proof

Since \( \eta \) be a t-symmetric around \( \left[ \frac{m_1^t + m_2^t}{2m_2^t} \right]^{\frac{1}{t}} \), using the definition \( \eta(a) = \eta \left[ \left[ m_1^t + m_2^t - a \right]^{\frac{1}{t}} \right] \)

\( \forall a \in [m_1^t, m_2^t] \) now set the variable \( s = m_1^t + m_2^t - a \)

Gives

\[
J^\eta_t(\eta(\phi))(m_2^t) = \frac{1}{n_{m_1^t}} \int_{m_1^t}^{m_2^t} (s - m_1^t)^{t-1} \eta \left( s^{\frac{1}{t}} \right) ds
\]

\[
= \frac{1}{n_{m_1^t}} \int_{m_1^t}^{m_2^t} (a - m_1^t)^{t-1} (m_2^t - a)^{t-1} \eta \left[ \left[ m_1^t + m_2^t - a \right]^{\frac{1}{t}} \right] da
\]

\[
= \frac{1}{n_{m_1^t}} \int_{m_1^t}^{m_2^t} (a - m_1^t)^{t-1} (m_2^t - a)^{t-1} \eta \left( a^{\frac{1}{t}} \right) da = J^\eta_t(\eta(\phi))(m_1^t)
\]

The proof is completed.

(ii) Proof is similar to (i).

Corollary 1

by the assumption of lemma 3

1. If \( t = 1 \) in (i) the result is

\[
J^\eta_1(\eta(m_2)) = J^\eta_1(\eta(m_1)) = \frac{1}{2} \left[ J^\eta_1(\eta(m_2)) + J^\eta_1(\eta(m_1)) \right]
\]

2. If \( t = -1 \) in (ii) then we have

\[
J^\eta_{-1}(\frac{1}{m_2}) = J^\eta_{-1}(\frac{1}{m_1}) = \frac{1}{2} \left[ J^\eta_{-1}(\frac{1}{m_2}) + J^\eta_{-1}(\frac{1}{m_1}) \right]
\]

Theorem 4 [15]

Let \( t \in R \setminus \{0\} \). If \( \eta : [m_1, m_2] \subseteq (0, \infty) \rightarrow R \) be a t-convex function with the inequality \( m_1 < m_2 \) and \( \eta \in [m_1, m_2] \)

\( \eta : [m_1, m_2] \subseteq R \setminus \{0\} \rightarrow R \) be a non-negative integrable and t-symmetric around the \( \left[ \frac{m_1^t + m_2^t}{2m_2^t} \right]^{\frac{1}{t}} \), then

a) For \( t > 0 \) have following inequality

\[
\eta \left( \left[ \frac{m_1^t + m_2^t}{2m_2^t} \right]^{\frac{1}{t}} \right) \left[ J^\eta_t(\eta(\phi))(m_2^t) + J^\eta_t(\eta(\phi))(m_1^t) \right]
\]
\[
\leq \left[ J^{\infty}_{\alpha} (\eta(\phi))(m^\prime) + J^{\infty}_{\alpha} (\eta(\phi))(m^\prime) \right]
\leq \frac{\eta(m^\prime) + \eta(m^\prime)}{2m^\prime}\left[ J^{\infty}_{\alpha} (\eta(\phi))(m^\prime) + J^{\infty}_{\alpha} (\eta(\phi))(m^\prime) \right]
\]
(28)

With \( o > 0 \) and \( \phi(a) = a^\gamma \forall a \in [m^\prime, m^\prime] \)

b) \( t < 0 \), following inequalities

\[
\eta \left( \frac{m^\prime + m^\prime}{2m^\prime} \right) \left[ J^{\infty}_{\alpha} (\eta(\phi))(m^\prime) + J^{\infty}_{\alpha} (\eta(\phi))(m^\prime) \right]
\]
\[
\leq \left[ J^{\infty}_{\alpha} (\eta(\phi))(m^\prime) + J^{\infty}_{\alpha} (\eta(\phi))(m^\prime) \right]
\leq \frac{\eta(m^\prime) + \eta(m^\prime)}{2m^\prime}\left[ J^{\infty}_{\alpha} (\eta(\phi))(m^\prime) + J^{\infty}_{\alpha} (\eta(\phi))(m^\prime) \right]
\]
(29)

Proof
\( \eta \) be a t-convex on \([m^\prime, m^\prime]\) then we have,

\[
\eta \left( \frac{a^\prime + b^\prime}{2} \right) \leq \frac{\eta(a) + \eta(b)}{2}
\]

Now take the values \( a^\prime = lm^\prime + (1-t)m^\prime \) and \( b^\prime = (1-t)m^\prime + lm^\prime \) for \( l \in (0,1) \), then we get,

\[
\eta \left( \frac{m^\prime + m^\prime}{2m^\prime} \right) \leq \frac{\eta[lm^\prime + (1-t)m^\prime] + \eta[(1-t)m^\prime + lm^\prime] + \eta[(1-t)m^\prime + lm^\prime]}{2}
\]
(30)

Multiplying this equation by \( \frac{1}{ln} l^\gamma (1-l)^{a-1} \eta[ln^\prime + (1-l)m^\prime] \) on both sides, \( o > 0 \) and then integrating about \( l \) over \([0,1]\), we obtain

\[
2 \left[ \frac{\eta(m^\prime + m^\prime)}{2m^\prime} \right] \int_0^1 l^\gamma (1-l)^{a-1} \eta[ln^\prime + (1-l)m^\prime] dl
\]
\[
\leq \frac{1}{ln} \int_0^1 l^\gamma (1-l)^{a-1} \eta[ln^\prime + (1-l)m^\prime] + \eta[(1-t)m^\prime + lm^\prime] dl
\]
\[
+ \frac{1}{ln} \int_0^1 l^\gamma (1-l)^{a-1} \eta[(1-t)m^\prime + lm^\prime] + \eta[(1-t)m^\prime + lm^\prime] dl
\]
(31)

Since \( \eta \) is a non-negative integrable and t-symmetric with respect to \( \left[ \frac{m^\prime + m^\prime}{2m^\prime} \right] \), then.

\[
\frac{\eta[(ln^\prime + (1-l)m^\prime)]}{\eta[(ln^\prime + (1-l)m^\prime)]} = \eta[ln^\prime + (1-t)m^\prime]
\]

Now choose the \( s = lm^\prime + (1-t)m^\prime \) then,

\[
\frac{2m^\prime}{ln(m^\prime - m^\prime)} \eta \left( \frac{m^\prime + m^\prime}{2m^\prime} \right)^{a-1} \int_{m^\prime}^m (m^\prime - s)^{a-1} \eta(s) ds
\]
\[
\leq \frac{1}{ln(m^\prime - m^\prime)} \left[ \int_{m^\prime}^m (m^\prime - s)^{a-1} \eta(s) ds \right] + \int_{m^\prime}^m (m^\prime - s)^{a-1} \eta[(m^\prime + m^\prime - s) \eta(s)] ds
\]

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\[
\frac{1}{|\ln (m'_2 - m'_1)|} \int_{m'_1}^{m'_2} (s - m'_1)^{\alpha -1} \eta (s^{\frac{\nu}{\nu}}) \eta (s^{\frac{\nu}{\nu}}) ds
\]
\[
+ \int_{m'_1}^{m'_2} (s - m'_1)(m'_2 - s) \eta (s^{\frac{\nu}{\nu}}) \eta (m'_2 + m'_1 - s) ds
\]
(32)

by the lemma2 we have
\[
\frac{m'_1}{(m'_2 - m'_1)^{\alpha}} \left[ \int_{m'_1}^{m'_2} \left( \frac{m'_1 + m'_2}{2m'_2} \right)^{\alpha} \eta (\eta (m'_1) + J_{\alpha}^{m'_2} (\eta (m'_1)) \right)
\]
\[
\leq \frac{1}{(m'_2 - m'_1)^{\alpha}} J_{\alpha}^{m'_2} (\eta (m'_2) + J_{\alpha}^{m'_2} (\eta (m'_2)))
\]
(33)

This result completes the inequality (28) and for the second inequality, if \( \eta \) is a t-convex function then we have,
\[
\eta \left( \left[ m'_1 + (1-t)m'_2 \right]^{\alpha} \right) + \eta \left( \left[ m'_2 + (1-t)m'_1 \right]^{\alpha} \right) \leq \left[ \eta (m'_1) + \eta (m'_2) \right]
\]
(34)

Multiplying by \( \frac{1}{n} \int (1-1)^{\alpha-1} \left[ \left[ m'_1 + (1-t)m'_2 \right]^{\alpha} \right) on both side of the equation (34) and integrating about \( l \)
over \([0,1]\) then we have
\[
\frac{1}{|n|} \int_{0}^{1} \left[ \int \left[ \left[ m'_1 + (1-t)m'_2 \right]^{\alpha} \right) dt
\]
\[
+ \frac{1}{|n|} \int_{0}^{1} \left[ \int \left[ \left[ m'_2 + (1-t)m'_1 \right]^{\alpha} \right) \right] dt
\]
\[
\leq \left[ \eta (m'_1) + \eta (m'_2) \right] \frac{1}{|n|} \int_{0}^{1} \left[ \int \left[ \left[ m'_1 + (1-t)m'_2 \right]^{\alpha} \right) dt
\]
(35)

That is
\[
\frac{1}{(m'_2 - m'_1)^{\alpha}} J_{\alpha}^{m'_2} (\eta (m'_2) + J_{\alpha}^{m'_2} (\eta (m'_2))),
\]
\[
\leq \frac{1}{(m'_2 - m'_1)^{\alpha}} \left[ \eta (m'_1) + \eta (m'_2) \right] \left[ J_{\alpha}^{m'_2} (\eta (m'_2)) + J_{\alpha}^{m'_2} (\eta (m'_2)) \right]
\]
(36)

The proof is complete.
(ii) proof is similar to of (i).

Corollary2
Under the assumption of theorem4
1. If \( t = 1 \) then
\[
\eta \left( \frac{m'_1 + m'_2}{2} \right) \left[ J_{\alpha}^{m'_2} (\eta (m'_2)) + J_{\alpha}^{m'_2} (\eta (m'_2)) \right] \leq \left[ J_{\alpha}^{m'_1} (\eta (m'_1)) + J_{\alpha}^{m'_1} (\eta (m'_1)) \right]
\]
\[
\leq \frac{\eta (m'_1) + \eta (m'_2)}{2} \left[ J_{\alpha}^{m'_1} (\eta (m'_1)) + J_{\alpha}^{m'_1} (\eta (m'_1)) \right]
\]
(37)

2. If \( t = -1 \) then
\[
\eta \left( \frac{2m'_1 + 2m'_2}{m'_1 + m'_2} \right) \left[ J_{\alpha}^{m'_2} (\phi (\frac{1}{m'_1})) + J_{\alpha}^{m'_2} (\phi (\frac{1}{m'_1})) \right]
\]
\[
\leq \left[ J_{\alpha}^{m'_2} (\phi (\frac{1}{m'_1})) + J_{\alpha}^{m'_2} (\phi (\frac{1}{m'_1})) \right]
\]
\[
\leq \frac{\eta (m'_1) + \eta (m'_2)}{2} \left[ J_{\alpha}^{m'_2} (\phi (\frac{1}{m'_1})) + J_{\alpha}^{m'_2} (\phi (\frac{1}{m'_1})) \right]
\]
(38)

Lemma3
Let \( t \in R \backslash \{0\} \) if \( \eta: [m_i, m_z] \subseteq (0, \infty) \rightarrow R \) be a differential function and \( \eta \in [m_i, m_z] \) if \( \eta: [m_i, m_z] \subseteq R \backslash \{0\} \rightarrow R \) be a non-negative integrable and t-symmetric around the \( \left[ \frac{m_i^l + m_i^u}{2m_z^u} \right] \) then

\[
\eta(m_i^l) + \eta(m_i^u) = \frac{1}{2} \left[ J_n^\eta \left( \eta(m_i^l) \right) + J_n^\eta \left( \eta(m_i^u) \right) \right] \]

\[
- [J_n^\eta \eta \left( \phi(m_i^l) \right) + J_n^\eta \eta \left( \phi(m_i^u) \right)]
\]

\[
\leq \frac{1}{n} \int_{m_i^l}^{m_i^u} \int (m_z^2 - y) \left( y - m_i^l \right)^{\alpha-1} \eta(\phi(y)) dy - \int_{x}^{m_z^2} \left( m_z^2 - y \right)^{\alpha-1} \eta(\phi(y)) dy
\]

\[
\eta \left( \phi(x) \right) dx
\]

With \( o > 0 \) and \( \phi(a) = a^{\frac{1}{\alpha}} \forall a \in [m_i^l, m_i^u] \)

\[
\eta(m_i^l) + \eta(m_i^u) = \frac{1}{2} \left[ J_n^\eta \left( \eta(m_i^l) \right) + J_n^\eta \left( \eta(m_i^u) \right) \right] \]

\[
- [J_n^\eta \eta \left( \phi(m_i^l) \right) + J_n^\eta \eta \left( \phi(m_i^u) \right)]
\]

\[
\leq \frac{1}{n} \int_{m_i^l}^{m_i^u} \int (m_z^2 - y) \left( y - m_i^l \right)^{\alpha-1} \eta(\phi(y)) dy - \int_{x}^{m_z^2} \left( m_z^2 - y \right)^{\alpha-1} \eta(\phi(y)) dy
\]

\[
\eta \left( \phi(x) \right) dx
\]

Where \( \phi(a) = a^{\frac{1}{\alpha}} \forall a \in [m_i^l, m_i^u] \)

Proof

a) \[
I = \int_{m_i^l}^{m_z^2} \int (m_z^2 - y) \left( y - m_i^l \right)^{\alpha-1} \eta(\phi(y)) dy \eta(\phi(x)) dx
\]

b) Integrating by part and using the result of Lemma3 we get

\[
I = \left[ \int (m_z^2 - y) \left( y - m_i^l \right)^{\alpha-1} \eta(\phi(y)) dy \right]_{m_i^l}^{m_i^u} \eta(\phi(x)) dx
\]

\[
= \left[ \eta \left( \phi(m_i^l) \right) + J_n^\eta \eta \left( \phi(m_i^l) \right) \right] - \left[ \eta \left( \phi(m_i^u) \right) + J_n^\eta \eta \left( \phi(m_i^u) \right) \right]
\]

Similarly

\[
I = \int_{x}^{m_z^2} \left( m_z^2 - y \right)^{\alpha-1} \eta(\phi(y)) dy \eta(\phi(x)) dx
\]

Integrating by part and using the result of Lemma3 we get

\[
I = \left[ \int (m_z^2 - y) \left( y - m_i^l \right)^{\alpha-1} \eta(\phi(y)) dy \right]_{m_i^l}^{m_z^2} \eta(\phi(x)) dx
\]

\[
= \left[ \eta \left( \phi(m_i^l) \right) + J_n^\eta \eta \left( \phi(m_i^l) \right) \right] - \left[ \eta \left( \phi(m_z^2) \right) + J_n^\eta \eta \left( \phi(m_z^2) \right) \right]
\]
\[+ \int_{\alpha}^{\beta} \left( x - m_i^\alpha \right) \left( m_i^\alpha - x \right)^{n-1} \eta(\phi(x)) \eta(\phi(x)) dx
\]

\[= \left[ \eta(\phi(m_i^\alpha)) \right]^{\alpha} \left[ J_{n-1}^{\alpha} \eta(\phi(m_i^\alpha)) \right]^{\alpha} \left[ J_{n-1}^{\alpha} \eta(\phi(m_i^\alpha)) \right]^{\alpha} \left[ \eta(\phi(m_i^\alpha)) \right]^{\alpha}
\]

\[= \left[ \frac{\eta(\phi(m_i^\alpha))}{2} \right] \left[ J_{n-1}^{\alpha} \eta(\phi(m_i^\alpha)) \right]^{\alpha} \left[ J_{n-1}^{\alpha} \eta(\phi(m_i^\alpha)) \right]^{\alpha} \left[ \eta(\phi(m_i^\alpha)) \right]^{\alpha}
\]

By the equation (42) and (43)

\[I^* = I_1 - I_2
\]

\[= \left[ \frac{\eta(m_1) + \eta(m_2)}{2} \right] \left[ J_{n-1}^{\alpha} \eta(\phi(m_1)) \right]^{\alpha} \left[ J_{n-1}^{\alpha} \eta(\phi(m_2)) \right]^{\alpha} \left[ \eta(\phi(m_1)) \right]^{\alpha}
\]

Multiplying the equation (44) by \( \frac{1}{n} \), then we get the equation (40)

(ii) proof is similar to that (i).

CONCLUSION

In this research paper we present the basic facts of HH and HH-f inequality for convex function and established some contribution of inequality theory and probability theory from the perspective of applications.

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