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Equilibrium analysis in game theory using nonlinear functional analysis techniques

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Abstract: - Game theory provides a mathematical framework for analyzing the tactical choices of rational decision makers. The idea of equilibrium, which states that no player has a motivation to change their existing approach given the strategies of other players, is one of the core ideas of game theory. This study examines a type of undirected graph-based distributed quadratic game. The problem of communication topology constraints is presented and nonlinear dynamics with uncertain time-dependent perturbations are present in the participant's dynamics. Based on a high-gain observer approach, a distributed Nash equilibrium (NE) finding technique is given, and the Lyapunov stability theory is used to study the convergence. It represents that every player approximates the positions of their rival players and that there are differences between the NE and the placements of the minor limitation that finally restricts the players. In addition, chattering problems are eliminated since the offered theory's formulation employs the hyperbolic tangent function to control the perturbation rather than the signum function. In an imitation of the oligopoly match, five enterprises manufacture identical materials in a duopoly market framework; this is done to confirm the effectiveness of the recommended strategy. Our results provide novel perspectives and methods for understanding complicated strategic situations, helping to close the gap between mathematical theory and real-world applications.

Keywords: Equilibrium analysis, Nash equilibrium, game theory, nonlinear functional analysis.

I. Introduction

Game theory's equilibrium analysis explores the strategic connections between decision-makers by utilizing nonlinear efficient analysis methods. The numerical foundation for model concerned, energetic systems where player strategies network is provided by nonlinear functional analysis [1]. This systematic method explores for equilibrium points while altering a player's strategy unilaterally results in a loss of advantage. It provides insight into possible dynamics and stable outcomes in environments that are either competitive or cooperative.

A. Game theory

A field of applied math called game theory is widely studied and used in an assortment of real-world situations, counting invest and trading. It provides logical methods for analyze scenario in which numerous people, referred as players, make related choices. When decide on a course of action in this situation, each person must take other people's potential decisions or plans into thought [2]. In doing so, a game solution outlines the best options for players with equivalent, at variance, or conflicting welfare as well as the many outcomes that can result from these choices. Game theory is useful in a wide range of scenario where choices made by individuals affect their outcomes. By emphasizing the strategic aspects of decision making by participants rather than random events, it enhances and expands the conventional theory of chance [3].

B. Equilibrium

The NE is a key theorem in game theory that influences decision-making, which states that a player can achieve their desired outcome by adhering to their starting strategy without deviation. Every player's approach in the NE is effective while considering other players' choices. All participants benefit as their desired results are achieved [4]. The definition of NE in current times refers to mixed strategies, during which participants select a probability distribution over potential pure approach.

C. Equilibrium analysis in game theory

NE is a tool used by game theorists to examine how several decision-makers interact strategically. Each decision-maker in a strategic interaction must consider both their own and the decisions of each other to attain an outcome [5]. Nash's concept is based on the straightforward realization that analyzing individual decision-makers actions in isolation fail to enable one to forecast their choices. Instead, it's necessary to consider what each player could accomplish if they were to assume that the other players would do the same approach [6]. Consistency in

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decision-making is necessary for Nash equilibrium since no player would want to change their choice in light of that is the other players are deciding.

D. Nonlinear functional analysis techniques

The smart area of nonlinear investigation in the twenty-first century is a compelling combination of numerical modeling, topology, nonlinear operative assumption, nonlinear serviceable analysis and real-world applications [7]. Its wide-ranging application encompasses anything from investigating the geometric characteristics of infinite-dimensional function spaces to solving practical problems in a variety of multidisciplinary fields. This covers disciplines including hydromechanics, astronomy, stochastic game theory, theoretical mechanics and the field [8].

Game theory provides a numerical structure to investigate strategic connections between normal decision-makers. The idea of equilibrium is crucial to assert that no performer has a motivation to change their plan in light of the devices of others. This theory is critical for sympathetic stable outcomes and decision-making processes in a variety of settings, including economics, political negotiations and evolutionary biology. It explains how those or entity make decisions in aggressive or cooperative contexts, ensuing in cost that affect not just themselves but also the larger system or humanity [9]. Game theory has become widely used in several fields of research through the past ten years, counting computer science, financial side, biology and more. The advancement of game theory has highlighted the significance of NE search in non-cooperative game, both in theory and in real-world applications [10]. Contributions of game theory to multi agent system manage embrace optical networks, smart grids, mobile sensor networks and more. The objective of this study investigates game theory NE through the application of nonlinear functional analysis methods. Particularly, the study offers a mathematical framework that can accurately represent and analyze strategic interactions between rational decision-makers.

E. Motivation of the study

This study is motivated by the difficulty of strategic interactions in game theory and the need for higher analytical techniques to identify and describe these dynamics with reliability [11]. This study employs nonlinear functional analysis techniques for NE analysis to offer a more thorough knowledge of rational agents' decision-making process. This approach is forced by the essential of bridging theoretical concepts with real-world application to offer a more nuanced understanding of planned behaviors and to assist in the development of strategies that are robust and adaptable in these conditions [12].

F. Contribution of the study

1. To analyze the strategic interactions among rational decision-makers using game theory as a mathematical framework.
2. To explore the implications and applications of a distributed quadratic game utilizing undirected graphs in network interactions.
3. To analyze the restrictions of communication topology and nonlinear dynamics with intermittent, with time-dependent disturbances in participant dynamics.
4. To enhance comprehension and applicability in strategic analysis by providing fresh perspectives and methods for interpreting complex strategic environments and bridging the gap between theoretical mathematics and real-world application.

II. RELATED WORKS

Game theory was impacted by the dissipative and passivity theories, particularly learning in online games that showed in [13] and the game theoretic field has put out a wide range of methods and dynamics over the years for determining the Nash equilibrium. Study [14] Nash Stackelberg Nash (NSN) games were a kind of two-stage hierarchy game with various leaders and followers. They specifically examine NSN games in the presence of Decision Dependent Uncertainties (DDUs). Game equilibrium analysis has seldom dealt with DDUs, or decision-maker uncertainties, that were influenced by the strategy. The history of generalized implicit vector equilibrium problems (GIVEP) and equilibrium problems (EP) was first provided in [15]. A selection of GIVEP existence theorems was presented, along with the necessary conditions for the GIVEP solution set to be convex and compact for set-valued mappings that were a subset of values in topological space [16]. In the non-cooperative scenario and even after a while, the participants' personal reward values were the only ones accessible [17], calculate the strong NE and suggest a method for establishing the Pareto frontier. The process used the Newton optimization approach to identify the strong NE that was advantageous for ill-conditioned problems. Study [18] used the Proof of Work (PoW) a consensus process to develop a pool of mining game play models and examine its NE from two angles.

Numerical simulations were used to examine the effects of mining pool power, infiltrated power percentage terms and deployment miner defection rate on the selection and revenue of the rate at which infiltration occurs. Adaptive graphics games on fixed communication graphs for linear multi-agent systems with a leader investigated in [19]. The goal was getting every agent to synchronize with the leader while simultaneously optimizing a performance index based on neighboring and own control rules. The related works summary regarding the equilibrium analysis in game theory is presented in Table 1.

Table 1: Output of the related work

Reference	Objective	Findings	Limitation
[20]	A differential game model for N people that was not cooperative.	Where each player tries to individually optimize their income by just keeping updated on power prices.	A differentiable game model becomes increasingly complex as the number of participants rises.
[21]	A generalized method for determining the ideal Bayesian-NE (BNE). To enable realistic network defense and assault.	Role-shifting in network attack and defensive contexts was taken into consideration by combining the NE issue with a reward maximization problem in the suggested method.	Complicated computations and generating reliable probabilistic evaluations were the reasons behind the limits of the BNE.
[22]	Established the limits on the empirical game theory examination of intricate relationships between several agents.	To offer many theoretical outcomes for empirical game theory.	Quantitative game theory has challenges while examining intricate relationships with various individuals, including finding a balance between accessibility and reliability.
[23]	Create a prospective game to find the non-cooperative game's NE with complete knowledge, then use the adaptive expectations approach to examine the consumer system dynamics in a situation with imperfect information.	Recognizing the aggregator's worth in terms of information and flexibility aggregation.	Meticulous planning and computing efficiency required powerful processing resources and sophisticated techniques for instantaneous analysis.
[24]	The authors proposed and solved a non-quadratic dynamic game model to explain the energy trade between prosumers and stable Nash equilibrium.	The framework was modeled as an infinite strategy, non-cooperative multiplayer game in which players have non-quadratic reward functions.	Possess restrictions such as greater computing load, discontinuity and difficulty.
[25]	To provide a transmission system expansion planning issue for the power sector based on cooperative game theory (CGT).	An emphasis on coordinated collaboration among players to improve the power grid.	Emotions, prejudices and cognitive constraints frequently cause human conduct to diverge from rigorous logic.

A. 2.1 Problem statement

Game theory deals with the strategic interactions between rational decision-makers. It is intended to look at situations when decisions made by one person affect those made by others. Because games involve players, strategies and rewards, they are used to illustrate these interactions. To maximize their utility, players take into account the methods chosen by others when making their own decisions. The study presents a high-gain observer strategy-based approach for calculating the distributed Nash equilibrium. It's shown that every player makes an assessment of the states of their competitors and that the variations between player positions and the NE are eventually limited by a small restriction.

III. Materials and Methods

The main purpose of the dispersed NE search problem is to design a distributed control strategy that allows players to find NE in a scenario with directs access to other players' states or cost functions. The conditions for

Lipschitz continuity, the existence of specific integers and stringent diagonally dominating matrices, respectively are given by assumptions 2, 3 and 4, ensuring the stability and uniqueness of solutions for Nash equilibrium. The study of convergence and uniqueness of the solution in the generalized NE search problem is based on these assumptions.

A. Preliminaries and Problem Formulation

1) Graph Theory

To study a graph with no direction$= (\mathcal{V}, \mathcal{E})$, where the edge set is $\varepsilon \subset v \times v$and the node set is $W = \{1, 2, \ldots, N\}$. It describes the information exchange inside a network system. Where $(i, j) \in \mathcal{E}$ denotes the possibility that player j can acquire anything from player i. Given the undirected nature of G^*, while $(i, j) \in \mathcal{E}$ consequently $(i, j) \in \mathcal{E}$ obtains. A route is made up of several edges having the pattern$(j, i), (i, l)$ .. . If every pair of nodes in an undirected network has a route linking them, the graph is said to be connected. Matrix of adjacency $B = [b_{ji}] \in \Re^{M \times M}$for graph $\mathcal{G}$ is established in a way that $b_{ji} = 0$alternatively and $b_{jj} = 0, b_{ji} > 0 \; if \; (i, j) \in E$. Assuming that $k_{jj} = \sum_{i=1}^{M} b_{ji}, \; and \; k_{ji} = -b_{ji} \; for \; any \; i \neq j$, the Laplacian matrix $K = [k_{ji}] \in \Re^{M \times M}$ is obtained.

Assumption 1: There is connectivity in Graph $\mathcal{G}$.

Remark 1: Assumption 1 is widely used in distributed games to assure that each additional player has access to every player's information.

Lemma 1: Assume that $B = diag\{b_{11}, \ldots, b_{1M}, b_{21}, \ldots, b_{MM}\}$in graph H and that K is its Laplacian matrix. Assume that point one is true. While this occurs, the matrix $(K \otimes J_{M \times M} + A)$ is positively definite.

Proof: Use $j = 1, \ldots, M$and let $A_J = diag\{b_{i1}, \ldots, b_{iM}\}$as the evidence. For that reason, $(K \otimes J_{M \times M} + A) = diag\{(K + A_j)\}$. One gets the idea from assumption 1 that for any matrix A_{jj}, there is at least one $b_{ji} > 0$. It needs to be noted that the connected, undirected graph G* has a Laplacian matrix. The matrix $K + A_j$ is symmetric positive definite. Therefore, the proof is finished and the matrix $diag\{(K + A_j)\}$ is positive definite.

2) Problem Formulation

Imagine a quadratic game with N players. Where$\mathcal{V} = \{1, \ldots, N\}$ indicates the set of players. These are the general nonlinear form values for each player's dynamics expressed as in equation (1).

$$\dot{y}_J = f_j(y) + \theta_j(s) + u_j, \;\; j \in \mathcal{V} \tag{1}$$

Where the player's action is represented by $y = col(y_1, \ldots, y_M)$, the known map that is nonlinear is $fj(y)*$, the unidentified phrase of disturbance is $\theta_j(s)$ and the states are represented by $y_j \in \Re$and$u_j \in \Re$. The presence of uncertainty and disruptions, deterioration, or forecasting mistakes are likely the causes of the time-varying disturbance component $\theta_j(s)$. To describe the information transfer in this game, utilize the undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The goal of the j^{th} player is to reduce its cost function. Where I_j is expressed in quadratic form in equation (2).

$$I_j(y) = \frac{1}{2} \sum_{i=1}^{M} \sum_{l=1}^{M} D_{il}^{j} y_i y_l + \sum_{i=1}^{M} d_i^j y_i + e^j, \qquad j \in \mathcal{V} \tag{2}$$

Where $D_{jj}^{j} < 0$ and $D_{il}^{j} = D_{li}^{j}$ for alli, j, and$l \in \mathcal{V}$, are constants and D_{il}^{j}, d_i^j, and e^j. For a N player game, a state profile $y^* = col(y_1^*, \ldots, y_M^*)$ is considered to be Nash equilibrium in equation (3).

$$I_j(y_1^*, \; . \; . \; . \; , y_j^*, \; . \; . \; . \; . \; , y_M^*) \leq I_j(y_1^*, \; . \; . \; . \; , y_j^*, \; . \; . \; . \; . \; , y_M^*), \quad j \in \mathcal{V} \tag{3}$$

By unilaterally altering its state, a player could further lower its associated cost function, resulting in exactly what is meant by Nash equilibrium.

3) Dispersed NE Search Issue

Examine games (1) and (2) for N players. The conditions and financial aspects of other participants aren't directly accessible to the j^{th}player. The neighbors of the j^{th} player is defined in Graph $\mathcal{G}$. A dispersed command method u_j for the j^{th}player was devised to determine the NE of games (1) and (2).

Remark 2: Equation (2) illustrates the cost functions that are quadratic, which are of significant importance in theory of games. Initially, it is used for various games. A generic model termed game (1) and (2) that used to illustrate a wide range of practical situations in which the quadratic cost function (2) has real-world implications. In the Oligopoly game for instance, (2) stands in for each company's profit. It also indicates the formation error in a formation control and engagements issue. Additionally, it serves as a second-order approximation for various nonlinear cost function classes. For the following analysis, a few presumptions are assumed.

Assumption 2: The Lipschitz constant K_f is used to globally Lipschitz the function$fj(y)$ $\forall j \in V$.

Assumption 3: Certain integers $\overline{\theta_j}$are positive and exist such that equation (4).

$$\left|\theta_j(s)\right| \leq \overline{\theta_j} \;\; \forall j \in \mathcal{V} \tag{4}$$

Assumption 4: Strictly diagonally dominating is the matrix$D$ expressed in equation (5).

$$D = \begin{pmatrix} D_{11}^1 D_{12}^1 & \cdots & D_{1M}^1 \\ D_{21}^2 D_{22}^2 & \cdots & D_{2M}^2 \\ \vdots & \ddots & \\ D_{M1}^M D_{M2}^M & \cdots & D_{MM}^N \end{pmatrix} \tag{5}$$

Remark 3: In the stability issue for a nonlinear system, assumptions 2 and 3 are frequently used. Every player's state converges throughout the world y_j to the relevant NE and it is sufficiently conditioned by them. Furthermore, a consensus-based method is used to establish a comparable Lipschitz condition for a distributed NE searching issue.

Remark 4: The reality and distinctiveness of the NE solution are ensured by assumption 4. Specifically claim that once simply in the event there exists y_j such that the games with cost function permit Nash equilibrium form equation (6).

$$D_{jj}^j y_j^* + \sum_{i \neq 1} D_{ji}^j y_i^* + d_j^j = 0, \qquad j \in \mathcal{V} \tag{6}$$

Furthermore, matrix D is implied to be nonsingular by assumption 4. Consequently, there is a singular Nash equilibrium in equation (7).

$$y^* = -D^{-1}d \tag{7}$$

Here $d = col(d_1^1, \dots, d_M^M)$.

IV. Result and discussion

By using distributed NE searching theory, an oligopoly game is analyzed by specifying the market structure, simulating market demand and figuring out profit. A game with five players is employed to verify the theory. A distributed observer strategy is proposed for obtaining the states and cost functions of other participants, ensuring convergence towards Nash equilibrium. Simulation results demonstrate the effectiveness and boundedness of observer error. However, nonlinear dynamics and communication topology render existing NE search methods useless.

In this part, to provide a distributed NE search approach based on distributed observers for the quadratic games (1) and (2). First, a preliminary lemma is put forward. For $\mathrm{y} = \Re$ and $\eta > 0$, we have that Lemma 2 expressed as in equation (8).

$$|\mathrm{y}| - \mathrm{y}\tanh\left(\frac{\mathrm{y}}{\eta}\right) \leq \mathrm{l}\eta \tag{8}$$

Where $\mathrm{l}\eta$ is a constant that, when applied to $\mathrm{l}\eta = \mathrm{c}^{-(\mathrm{l}+1)}$, satisfies l = 0.2785. Assuming $\mathrm{I_i}$and Γ_{ik} are all in$\mathcal{V}$, define $\Gamma_{\mathrm{ik}} = \left(\frac{[\partial \mathrm{I_i}(x)]}{\partial \mathrm{y_i}}\right)\left(\frac{[\partial^2 \mathrm{I_i}(\mathrm{y})]}{[\partial \mathrm{y_i} \mathrm{y_k}]}\right)and\Gamma_{\mathrm{j}} = \sum_{\mathrm{i}=1}^{\mathrm{M}} \Gamma_{\mathrm{ij}}$. Take note that player *j* is the only one with access to the i^{th}player to search for the Nash equilibrium, function Γ_{j}, and state, $i \in \mathcal{V}$are required since system (1) contains the nonlinear map $\mathrm{f_j}(\mathrm{y})$ and the perturbation term $\theta_{\mathrm{j}}(s)$. Thus, they will create a distributed observer. Player j's estimates for function Γ_{ik} and state $\mathrm{y_i}$ are denoted by $\hat{}\Gamma_{\mathrm{ik}}$, Γ_{j} and $\mathrm{y_i}$ state. Consider system (1) with strategy from equation (9) to (10).

$$\dot{\widehat{\Gamma_{\mathrm{jik}}}} = -\delta^{-1}\left(\sum_{\mathrm{l}=1}^{\mathrm{M}} \mathrm{b_{jl}}(\hat{\Gamma}_{\mathrm{jik}} - \hat{\Gamma}_{\mathrm{lik}}) + \mathrm{b_{ji}}(\hat{\Gamma}_{\mathrm{jik}} - \hat{\Gamma}_{\mathrm{ik}})\right) \tag{9}$$

$$\dot{\widehat{\mathrm{y_{ji}}}} = -\delta^{-1}\left(\sum_{\mathrm{l}=1}^{\mathrm{M}} \mathrm{b_{jl}}(\hat{\Upsilon}_{\mathrm{ji}} - \hat{\mathrm{y}}_{\mathrm{lk}}) + \mathrm{b_{ji}}(\hat{\mathrm{y}}_{\mathrm{ji}} - \hat{\mathrm{y}}_{\mathrm{i}})\right) \tag{10}$$

$$\mathrm{u_i} = -\mathrm{f_j}(\widehat{\mathrm{y_i}}) - \alpha_{\mathrm{j}}\hat{\Gamma}_{\mathrm{j}} - \beta_{\mathrm{j}}\tanh\left(\frac{\hat{\Gamma}_{\mathrm{j}}}{\eta}\right) \tag{11}$$

Here $\mathrm{j, i, k} \in \mathcal{V}, \hat{\Upsilon}_{\mathrm{j}} = \mathrm{col}(\hat{\Upsilon}_{\mathrm{jk}}, \ldots, \hat{\Upsilon}_{\mathrm{jM}})$, $\hat{\Gamma}_{\mathrm{j}} = \sum_{\mathrm{i}=1}^{\mathrm{M}} \hat{\Gamma}_{\mathrm{jij}}$with α_0 and β_0 as a few positive constants and δ as the desired positive parameter.

Theorem 1: Assuming that premises 1-4 are true. In light of approach (4, 10, and 11), consider system (2). Then, for any $\alpha_{\mathrm{j}} \geq \alpha_{\mathrm{j},}\beta_0 \geq \beta_{\mathrm{j}} \geq \theta_{\mathrm{j}}$and $\eta > 0$, there are positive constantsδ^*, α_0^* ands_0 from equation (12).

$$\|y - y^{*}\| \le q_0 \forall s \ge s_0 \tag{12}$$

Using s_0 as the border. Furthermore, the convergence barrier s_0 can be adjusted to any small value by either rising or reducing parameter αi can at the moment provide the proof of Theorem 1. Evaluate (13-14).$\bar{\Gamma}_{jik}(s)\hat{\Gamma}_{jik}(s) - \Gamma_{ik}(s)$ and $\bar{\Gamma}_k$ $\mathrm{col}(\bar{\Gamma}_{11k}, \dots, \bar{\Gamma}_{1Mk}, \bar{\Gamma}_{21k}, \dots, \bar{\Gamma}_{MMk})$, j, i, l $\in \mathcal{V}$. Then there is that equation (13).

$$\dot{\bar{\Gamma}}_k = -\frac{1}{\delta}(L \otimes J_{M\times M} + A)\bar{\Gamma}_k - J_M \otimes \varphi k \tag{13}$$

Where φk is equal to $\mathrm{col}(\dot{\Gamma_{1k}}, \dots, \dot{\Gamma}_{1Mk})$. Examine $\bar{y}_{ji}(s) = \hat{y}_{ji}(s) y_i(s) j, i \in \mathcal{V}$, and $\bar{y} = \mathrm{col}(\bar{y}_{11}, \dots, \bar{y}_{1M}, \bar{y}_{21}, \dots, \bar{y}_{MM},)$are defined. Then there is equation (14),

$$\dot{\bar{y}} = -\frac{1}{\delta}(L \otimes J_{M\times M} + A)\bar{y} - J_M \otimes \phi \tag{14}$$

In Lemma 1, A is defined, L is graph G Laplacian matrix and $\tilde{y} = \mathrm{col}(\tilde{y}_1, \dots, \tilde{y}_M$. Take (11) into consideration, for each j in possible to rewrite system (2) using strategy (11) expressed as equation (15) to (17).

$$\dot{\tilde{y}}_j = f_j(y) - f_j(\hat{y}_j) - \alpha_j \hat{\Gamma}_j - \beta_j \tanh\left(\frac{\hat{\Gamma}_j}{\eta}\right) + \theta_j(s) = -\alpha_j \Gamma_j - \beta_j \tanh\left(\frac{\Gamma_j}{\eta}\right) + \theta_j(s) + c_j \tag{15}$$

Where,

$$c_j = f_j(y) - f_j(\hat{y}_j) + \alpha_j \Gamma_j + \beta_j tanh\left(\frac{\Gamma_j}{\eta}\right) - \left(\alpha_j \hat{\Gamma}_j + \beta_j tanh\left(\frac{\hat{\Gamma}_j}{\eta}\right)\right) \tag{16}$$

Then there is that,

$$\dot{\tilde{y}}_j = -\alpha_j \Gamma_j - \beta_j \tanh\left(\frac{\Gamma_j}{\eta}\right) + \theta_j(s) + c_j \tag{17}$$

Remark 5: The coordinate transformation $\tilde{y}_j = y_j - y_j^{*}$ where y_j^{*} represents the Nash equilibrium, is how the system (17) is formed. Thus, the challenge of obtaining NE is transformed into the system (17). Stabilization problem, the stabilization issue of system (17) can be solved by using the signum function to handle the disturbance $\theta_j(s)$. Lemma 2 states that the hyperbolic tangent function is continuous and behaves similarly to the signum function found in formula (16) for the sliding form control.

Remark 6: The coordinate transformations are used to obtain systems (13) and (14) which should be noted $\bar{\Gamma}_{jik}(s) = \hat{\Gamma}_{jik}(s) - \Gamma_{ik}(s)$ and $\bar{y}_{ji}(s) = \hat{y}_{ji}(s) - y_i(s)$. The observers (17) and (18) will thus acquire the states and cost functions of other players assuming systems (13) and (14) are stable.

Proposition 1: There is a constant γ that exists for systems (13), (14) and (17).

$$\left\|col(J_M \otimes \varphi l, J_M \otimes \phi, c_j\right\| \le \gamma \left\|col(\tilde{y}, \bar{y}, \bar{\Gamma}, \bar{\theta}\right\| \tag{18}$$

Here$\bar{\theta} = col(\overline{\theta_1}, \dots, \bar{\theta}_M), \bar{\Gamma} = col(\bar{\Gamma}_i, \dots, \bar{\ } N)\ with\ \bar{\ } i = i - \hat{\ } i, i \in V$.

The proof involves using $K_\Gamma =$ max $\{\alpha_j + (\beta_j/\eta\}$to represent the Lipschitz constant for the function $(\alpha_j \Gamma_j + (\beta_j\ tanh(\Gamma_j/\eta))$about$\Gamma_j$. Assumption 2 implies that functions f_i(x) have a global Lipschitz constant; they have equation (19),

$$\frac{\partial I_l}{\partial y_l} = D_{LL}^{L} y_L + \sum_{i \ne l} D_{li}^{l} y_i + d_l^{l}, \quad \Gamma_{il} = D_{ik}^{i} \frac{\partial I_l}{\partial y_l}(y) \tag{19}$$

Assume that $\Gamma = col(\Gamma_1, \dots, \Gamma_M) = col(\sum_{i=1}^{M} \Gamma_{i1}, \dots, \sum_{i=1}^{M} \Gamma_{iM})$ consequently, according to Remark 4, where is the unique Nash equilibrium, develops in equation (20).

$$\Gamma = D^S(Dx + d - Dy^{*} - d) = D^S D \tilde{y} \tag{20}$$

Observe that $\|\varphi_k\| \le \|\dot{\Gamma}\|$as equation (21) they have that:

$$\begin{aligned} \|I_M \otimes \varphi_k\| \le M\|\dot{\Gamma}\| \le M\lambda_{max}(D^S D)\|\tilde{y}\| \le MK_\Gamma \lambda_{max}^2(D^S D) + M^2 K_f \lambda_{max}(D^S D)\|\bar{y}\| \\ + M^2 K_\Gamma \lambda_{max}(D^S D)\|\bar{\Gamma}\| + M\lambda_{max}(D^S D)\|\bar{\theta}\| \end{aligned} \tag{21}$$

To construct the last inequality in (22) one uses the knowledge that,

$$\sum_{j=1}^{M} \left\|\alpha_j \Gamma_j + \beta_j \left(\frac{\Gamma_j}{\eta}\right)\right\| \le K_\Gamma \|\Gamma\| \tag{22}$$

By (23) they have that,

$$\|I_M \otimes \phi\| \le M\|\phi\| \le MK_\Gamma \lambda_{max}(D^S D)\|\tilde{y}\| + M^2 K_f \|\bar{y}\| + M^2 K_\Gamma \|\bar{\Gamma}\| + M\|\bar{\theta}\| \tag{23}$$

Using (14), the last inequality in (15) is also obtained. Thus, by using (23), (24) and (25), (26) is produced and γ is dependent on constants matrix M.

Proof of theorem 1: By demonstrating of closed-loop systems (27) and (28), they can currently demonstrate Theorem 1. By using Remark 4, one can easily determine the existence and uniqueness of the Nash equilibrium solution. Assumption 1 suggests that matrix $K\otimes J_{M\times M}+A$ is positive definite based on Lemma 1. Let O has the property that, as a symmetric positive-definite matrix,

$$O(K\otimes J_{M\times M}+A)+(K\otimes J_{M\times M}+A)^S O=J$$

$$V_2=(1/2)\,\bar{y}^S O\bar{y}$$

$$\dot{V}_2=-\frac{1}{\delta}\bar{y}^S\bar{y}\ +\bar{y}^S O(J_M\otimes\emptyset)$$
$$\leq -\frac{1}{\delta}\|\bar{y}\|^2+\lambda_{max}(O)\|\bar{y}\|\|J_M\otimes\emptyset\|$$
$$\leq -\left(\frac{1}{\delta}-\lambda_{max}(P)M^2K_f\right)\|\bar{y}\|^2$$
$$+\lambda_{max}(P)NL_\Gamma\lambda_{max}^2(D^SD)\|\bar{y}\|\|\bar{y}\|$$
$$+\lambda_{max}(P)N^2L_\Gamma\lambda_{max}(D^SD)\|\bar{y}\|\|\bar{\Gamma}\| \tag{24}$$

Let $\sum_{k=1}^{M}\left(-\frac{1}{\delta}\bar{\Gamma}_k^S O(J_M\otimes\varphi_k)\right)$. It follows from (24) that:

$$V_3=(1/2)\sum_{J=1}^{M}\bar{\Gamma}_k^S O\ \bar{\Gamma}_k$$

$$\dot{V}_3=\sum_{k=1}^{M}\left(-\frac{1}{\delta}\bar{\Gamma}_k^S O(J_M\otimes\varphi_k)\right)$$
$$\leq -\frac{1}{\delta}\|\bar{\Gamma}\|^2+\lambda_{max}(O)\sum_{k=1}^{M}\bar{\Gamma}_k^S(J_M\otimes\varphi_k)$$
$$\leq -\frac{1}{\delta}\|\bar{\Gamma}\|^2+\lambda_{max}(O)M\|\bar{\Gamma}\|$$
$$\times(NL_\Gamma\lambda_{max}^2(D^SD)\|\tilde{y}\|+M^2K_f\lambda_{max}(D^SD)\|\tilde{y}\|$$
$$+M^2K_f\lambda_{max}(D^SD)\|\Gamma\|+M\lambda_{max}(D^SD)\left\|\bar{\theta}\right\|$$
$$\leq -(\frac{1}{\delta}-\lambda_{max}(O)M^3\lambda_{max}(D^SD)K_\Gamma\|\bar{\Gamma}\|^2$$
$$+\lambda_{max}(O)M^2\lambda^2{}_{max}(D^SD)K_\Gamma\|\bar{\Gamma}\|\ \|\tilde{y}\|$$
$$+\lambda_{max}(O)M^3\lambda_{\ max}(D^SD)K_{\mathrm{f}}\|\bar{\Gamma}\|\ \|\tilde{y}\|$$
$$+\lambda_{max}(O)M^2\lambda_{\ max}(D^SD)\|\bar{\Gamma}\|\ \left\|\bar{\theta}\right\| \tag{25}$$

Assume that $U=U_1+U_2+U_3 y=col(\|\tilde{y}\|,\|\bar{y}\|,\|\bar{\Gamma}\|$. We have that based on (24) and (25).

$$\acute{U}\leq -z^S Lz+\mu z+M\beta_0 L\eta \tag{26}$$

Where $\mu=\lambda_{\ max}(O)M^2\lambda_{\ max}(D^S\ D)\left\|\bar{\theta}\right\|$.

And$L$ is a symmetric matrix as:

$$L=\begin{pmatrix}L_{11}L_{21}L_{31}\\L_{21}L_{22}L_{32}\\L_{31}L_{32}L_{33}\end{pmatrix} \tag{27}$$

With,

$$L_{11}=\propto_0\ \lambda_{min}^2(D^S\ D)$$
$$L_{11}=\frac{1}{\delta}-\lambda_{max}(O)M^2K_f$$
$$L_{33}=\frac{1}{\delta}-\lambda_{max}(O)M^3\lambda_{max}(D^SD)L_\Gamma$$
$$L_{21}=-\frac{M\lambda_{max}^2(D^SD)K_\Gamma+\lambda_{max}(O)MK_\Gamma\lambda_{max}^2(D^SD)}{2}$$
$$L_{31}=-\frac{M\lambda_{max}^2(D^SD)K_\Gamma+\lambda_{max}(O)M^2K\lambda_{max}^2(D^SD)L_\Gamma}{2}$$
$$L_{32}=-\frac{1}{2}\left(\lambda_{max}(O)M^3L_\Gamma\lambda_{max}(D^SD)+\lambda_{max}(O)M^3L_\Gamma\lambda_{max}(D^SD)K_f\right) \tag{28}$$

Furthermore, by increasing the parameters $\propto_0$ and decreasing δ, they can select L as a strictly diagonally dominating matrix. After that, matrix L is positive definite for any $\delta\ \in\ (0,\delta^*)$and $\alpha_0>.\ \alpha_0^*$ This is because there are constants $\delta^*$and α_0^*. Thus, it follows that $0\ <\sigma\ <\ 1$ exists such that equation (29):

$$\acute{U}\leq -\lambda_{min}(L)\|y\|^2+\mu\|z\|+N\beta_0 r\eta$$
$$=-(1-\sigma)\lambda_{min}(L)\|z\|^2-(\sigma\lambda_{min}(L)\|z\|^2-\mu\|z\|-M\beta_0 r\eta)$$
$$\leq -(1-\sigma)\lambda_{min}(L)\|z\|^2\forall\|z\|\geq q_0^* \tag{29}$$

As $q_0^* = (\mu + \frac{\sqrt{\mu^2+4\sigma\lambda_{min}(L)M\beta_0 r\eta)}}{2\sigma\lambda_{min}(L)}$, is the formula for according to Remark 4. After that using equation (30),

$$\varepsilon_1\|z\|^2 \le U \le \varepsilon_2\|z\|^2 \tag{30}$$

In which $\varepsilon_1 = min\{(1/2)\lambda_{min}(O), (1/2)\lambda_{min}(D^S D)\}$ and $\varepsilon_2/\varepsilon_1 = max\{(1/2)\lambda_{max}(O), (1/2)\lambda_{max}(D^S D)\}$ are found. Exists with equation (31):

$$\|z\| \le \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} q_0^* \forall s \ge s_0 \tag{31}$$

Assume that $q_0 = \sqrt{(\varepsilon_2/\varepsilon_1)}\, q_0^*$. Based on $\|y - u^*\| \le \|z\|$there exists t_0 such that equation (32).

$$\|y - u^*\| \le q_0 \ \forall t \ge s_0^* \tag{32}$$

Remember that by raising s_0 and reducing. Hence, by raising parameter s_0^* and reducing parameter δ, the convergence border q_0 could be set arbitrarily small.

Remark 7: The high-gain observer concept is borrowed into the design of the distributed algorithm (4). Specifically, $\delta - 1$ and α_j are selected to be big enough such that player j estimate follows player I state quickly and matrix L is strictly diagonally dominating. The constants and the matrices $P\ and\ D$ determine the lower boundaries δ^* andα_0^*. Here the starting time is denoted by s_0^*. Therefore system (2) and strategy (4) are integrating more quickly at the moment. With Theorem 1, they will show how simple it is to construct the distributed NE search approach to N-player no-cooperative game with an unpredictable temporally variable disturbance.

Corollary 1: Think about the next single-player N-player game.

Using the cost function in equation (34), the disseminated NE, where j provides a few uplifting constants and δ is a positive parameter. Furthermore, by raising or lowering parameterδ, one can control the convergence border q_1 to an arbitrarily small value.

Proof: The following proof, which is skipped here, is based on Theorem 1. They also demonstrate that Theorem 1 can be quickly extended to derive the standard NE-seeking procedure for quadratic games with unknown time-varying disturbance. Corollary 2: Consider for this time that premises 3 and 4 are accurate. Think about system (2) and its methodology as provided in equation (33).

$$u_j = -f_j(y) - \alpha_j\Gamma_j - \beta_j tanh\left(\frac{\Gamma_j}{\eta}\right), j \epsilon \nu$$

Subsequently, s_2 and $q_2(\eta)$occur such that equation (34).

$$\|y - y^2\| \le q_2(\eta) \quad \forall s \ge \ s_2 \tag{34}$$

Furthermore, by reducing parameter η, the convergence boundary$q_2(\eta)$ could be set indefinitely narrow. In instance, let $\eta \ = \ c^{-\eta 0 s}$ with$\eta_0 > \ 0$, then $limt \to \infty(y(s) \ - \ y^*) \ = \ 0.$

Proof: System (2) and method (23) can be modified as equation (35).

$$\dot{y}_j = \ -\alpha_j\Gamma_j - \beta_j tanh\left(\frac{\Gamma_j}{\eta}\right) + \theta_j(s) \tag{35}$$

Comparable to (9), they have that expressed equation (36).

$$\dot{\tilde{y}}_j = \ -\alpha_j\Gamma_j - \beta_j tanh\left(\frac{\Gamma_j}{\eta}\right) + \theta_j(s) \tag{36}$$

$V = (1/2)\sum_{i=1}^{M}\left([\partial I_i/\partial y_i](y)\right)^2$is defined.2. Take note that (12) suggests:

$$\lambda_{min}(D^S D)\|\tilde{y}\| \le \|\Gamma\| \le \lambda_{max}(D^S D)\|\tilde{y}\| \tag{37}$$

By (37) to (40), they have that,

$$\dot{V} \le -\alpha_0\|\Gamma\|^2 + \mathrm{N}\beta_0 k\eta \le -\alpha_0\lambda_{min}^2(D^S D)(\|\tilde{y}\|^2 - (q_2^*)^2) \tag{38}$$

Where:

$$q_2^* = \sqrt{\frac{(\mathrm{N}\beta_0 k\eta)}{\left(\alpha_0\lambda_{min}^2(D^S D)\right)}} \tag{39}$$

According to (3), they have that:

$$V = \frac{1}{2}\|(Dx + d\|^2 = \frac{1}{2}\|D\tilde{x}\|^2 \tag{40}$$

Following that, $\varepsilon_3\|\tilde{y}\|^2 \le \ V \ \le \ \varepsilon_4\|\tilde{y}\|^2$where $\varepsilon_3 \ = (1/2)\lambda_{min}(D^S D)and\varepsilon_4 \ = (1/2)\ \lambda_{max}(D^S D)$. Let$q_2 = \sqrt{(\varepsilon_4/\varepsilon_3)}q_2^*$. It follows that there exists s_2 such that equation (41),

$$\|y - y^2\| \le q_2 \quad \forall s \ge \ s_2 \tag{41}$$

According to equation (25), reducing parameter η allows for the tuning of the convergence border $q_2(\eta)$ to an arbitrarily tiny value. Let$\eta = c^{-\eta 0s}$with in particular, when$\eta_0 > 0$, next up, there expresses as in equation (42),

$$\dot{V} \leq -\alpha_0\lambda_{min}^2(D^SD)\|\tilde{y}\|^2 + N\beta_0 k\eta^{-\eta 0s} \tag{42}$$

It suggests that V is within$\mathcal{L}_\infty$. Given (26), we can conclude that$V \in \mathcal{L}_\infty$. From (24) it is evident that$\tilde{\tilde{y}} \in \mathcal{L}_2$. It follows that $lim_{\to\infty}\tilde{y}(s) = 0$ by Barbalat Lemma. This also suggests that$lim_{\to\infty}(y(s) - y^* = 0$.

A. *Oligopoly game theory*

The market structure of an oligopoly is a small number of businesses manufacture a uniform good and each firm is aware of its markets, which is usually examined with oligopoly game theory. Examining a five-player game, served as validation for the distributed NE searching theory. In a duopoly market structure, for instance, take five participants, or five companies manufacturing the same goods. It is assumed that the product's price never changes whilst playing the game. The market demand for their items $(r_j, j = 1, \ldots, 5)$ is provided by equation (43).

$$r_i = r_0 + \epsilon_0\left(\omega_j^2 - \frac{1}{5}\sum_{i=1}^{5}\omega_i^2\right) + \epsilon_j\omega_j \tag{43}$$

Wherer_0 represents the degree of product diversification, ω_j represents the fundamental market demand and $\epsilon_0 > 0$ and $\epsilon_j > 0$ represent the criteria for demand differentiation. Furthermore, $k_j, j = 1, \ldots$, 5 are modeled by equation (44).

$$\dot{\omega}r_i = -\frac{\gamma_1}{5}\sum_{i=1}^{5}\omega_j + \gamma_2(e_0(s) + e_j \tag{44}$$

Where the input benefit coefficient $\gamma_1 > 0$, the basic production cost (c0 (t) 3*), the slowly increasing cost $\gamma_2 > 0$ of the degree of differentiation are all present. Here $I_j = (O_i - n_i)r_j$represents each company's profit, denoting the product's pricing and marginal cost respectively.

$$l_i = -\frac{\gamma_1}{5}\sum_{i=1}^{5} l_j + \gamma_2(e_0(s) + \gamma_2 e_j$$

$$I_j = (O_i - n_i)\left(r_0 + \epsilon_0\left(\omega_j^2 - \frac{1}{5}\sum_{i=1}^{5}\omega_i^2\right) + \epsilon_j\omega_j\right) \tag{45}$$

Actually, (45) creates a dynamic mapping between the actions (O_i) of the players and the corresponding payment functions (I_j). Consider as the disturbance due to its gradual changes and difficulty in determining in practical applications. Let the additional expense Þ2ci* represent the corporation activity. The goal of the game is to create a variable control approach O_i for the ith* corporation to find the NE of the game. Game is thus represented by (2) and (3). Figure 1 provides the communication topology; the proof of assumption 1's validity is not difficult. The cost function is quadratic in game (45), whereas the map is linear. Assumptions 2-4 are true and possibly thus be easily verified.

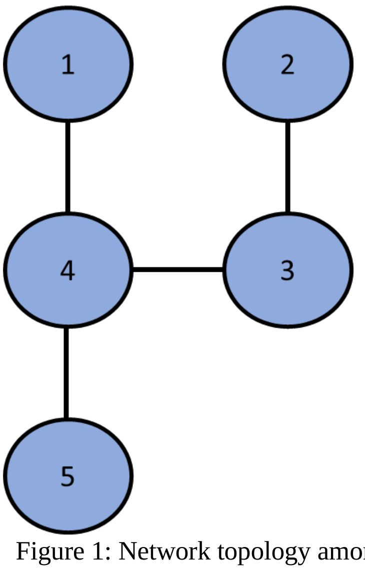


Figure 1: Network topology among players

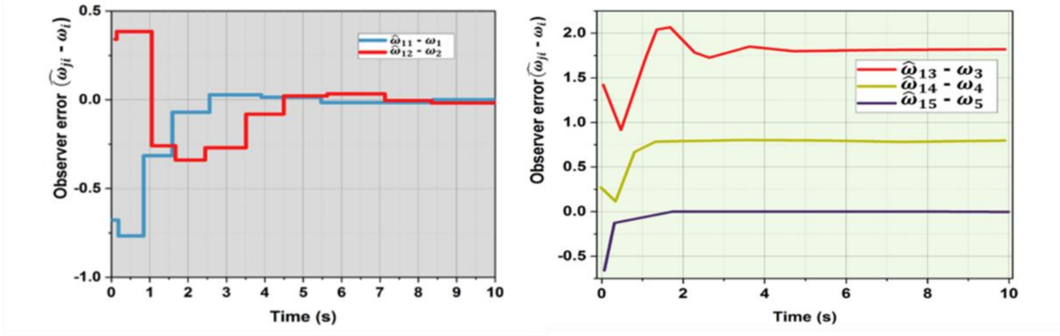
Assume that $\omega = col(\{\omega_1, \ldots, \omega_5\}$. Let $\omega *= col(\{\omega_1^*, \omega_2^*, \ldots, \omega_5^*\}) = [1, 1.4, 1.2, 1.6, 0.8]^T$ by Remark 4. The distinct NE is T. Next, the algorithm for searching is presented as:

$$\dot{\widehat{\Gamma_{\jmath\imath k}}} = -\delta^{-1}\left(\sum_{l=1}^{M} b_{jl}(\hat{\Gamma}_{jik} - \hat{\Gamma}_{lik}) + \mathrm{b}_{ji}(\hat{\Gamma}_{jik} - \hat{\Gamma}_{ik})\right) \tag{46 - a}$$

$$\dot{\widehat{\omega_{\jmath\imath}}} = -\delta^{-1}\left(\sum_{l=1}^{M} b_{jl}(\widehat{\omega}_{ji} - \widehat{\omega}_{lk}) + \mathrm{b}_{ji}(\widehat{\omega}_{ji} - \widehat{\omega}_{i})\right) \tag{46 - b}$$

$$e_i = -f_j(\widehat{\omega_\imath}) - \alpha_j\hat{\Gamma}_j - \beta_j tanh\left(\frac{\hat{\Gamma}_j}{\eta}\right) \tag{46 - c}$$

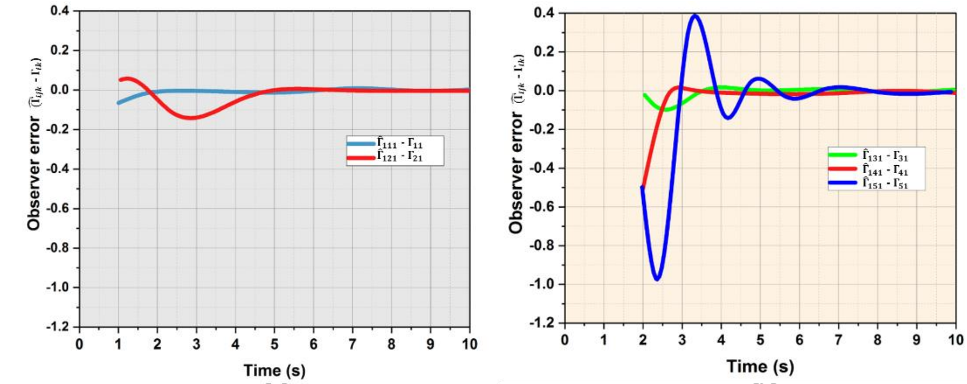
Remark 6 and Theorem 1 provide the distribution spectator in (29) with the states and cost functions of the other players, accordingly. The observer errors produced by algorithms (36a) and (36b) are shown in Figures 2 (a) and (b) as well as 3 (a) and (b).



(a) **(b)**

Figure 2: $\widehat{\boldsymbol{\omega}}_{\boldsymbol{ji}} - \boldsymbol{\omega}_{\boldsymbol{i}}$ with $\boldsymbol{j} = \mathbf{1}$ (Observer error)

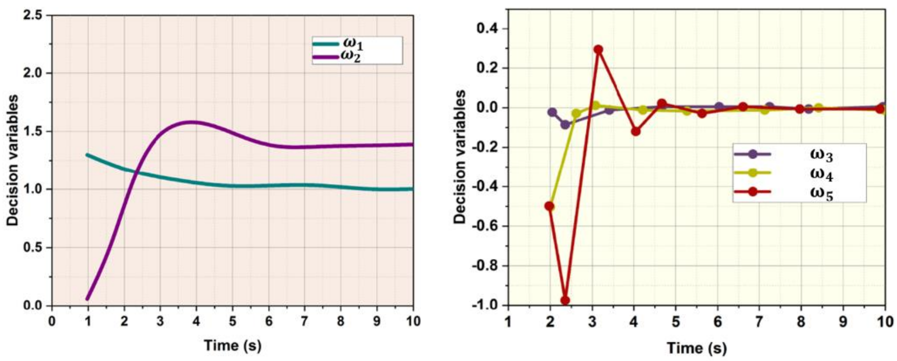
The simulation findings demonstrate the boundedness of the observer errors. Furthermore, by raising parameter α_0^* and lowering parameter δ^*, the observer boundary can be set arbitrarily tiny. Space constraints prevent the inclusion of the simulation figure 3 (a) and (b).



[a] [b]

Figure 3: $\hat{\boldsymbol{\Gamma}}_{\boldsymbol{ijk}} - \boldsymbol{\Gamma}_{\boldsymbol{ik}}$with i = 1 (observer error)

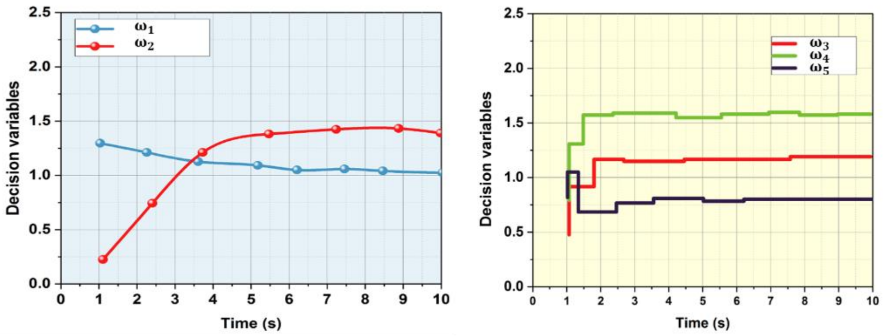
Figure 4 (a) and (b) display every state that each player has created using the searching algorithm (36). All of the player states converge to the distinct NE with a narrow border, as can be seen in the image. This attests to the suggested method's efficacy.



(a) **(b)**

Figure 4: State $\boldsymbol{\omega}_{\boldsymbol{i}}$ with $\boldsymbol{i} = \mathbf{1}, \ldots, \mathbf{5}$

It should be noted that game (35) is insoluble using the current solutions on the NE search issue. There are two primary causes. First, the communication topology limitation makes the conventional NE search technique inapplicable. For instance, since the game (35) necessitates that every Players' states and actions should be visible to all other players, the strategy selection in a symmetric oligopoly game is inapplicable. Second, due to game (35)'s nonlinear dynamics, the distributed NE searching strategies presently in use are similarly inapplicable, specifically the states of every player produced by the consensus-based method shown in Figure 5 (a) and (b). As a result, $\boldsymbol{\omega_i}$ does not converge to the Nash equilibrium and games cannot benefit from the consensus-based method (35).



(a) (b)

Figure 5: A consensus-based approach with state $\boldsymbol{\omega_i}$

V. CONCLUSION

A mathematical framework referred to as game theory is used to examine the strategic interactions among rational decision-makers. This study presents a comprehensive analysis of strategic interactions among logical decision-makers using game theory, focusing on distributed quadratic games represented by undirected graphs. It addresses communication topology constraints and nonlinear dynamics with uncertain time-dependent perturbations within player's strategies. By employing a high gain observer approach and Lyapunov stability theory, a distributed NE finding technique is proposed, ensuring convergence. Notably, the utilization of the hyperbolic tangent function effectively manages perturbations, mitigating chattering concerns. In other situations, such as the oligopoly games in China's broadband access marketing, they demonstrated that the player's interaction with the system is exponential. Through simulation of a duopoly market structure involving five companies producing identical commodities, the efficacy of the suggested strategy is validated. These findings offer novel perspectives and methodologies for navigating complex strategic scenarios, bridging the divide between theoretical frameworks and practical applications. Overall, this study contributes significantly to the understanding and management of strategic interactions in real-world contexts, providing valuable insights for decision-makers across various domains.

A. Limitation

The limitations derive from its assumptions on continuity, differentiability and convexity, which are not applicable in real-world scenarios. The complexity of games and the challenge of capturing their dynamic aspects are further barriers.

B. Future Research

The use of nonlinear functional analysis methods in game theory will greatly improve equilibrium analysis research in the future. This method improves one's capacity for making strategic decisions by offering a more thorough grasp of evolutionary game dynamics, mixed tactics and non-cooperative games.

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