J. Electrical Systems 20-4s (2024):2139-2149

[1]Mehdi Nadjafikhah

[2]Hind Al-Bderi

A note on Lie symmetry analysis, optimal system, new solitary wave solutions and conservation laws of the Pavlov equation

Journal of Electrical Systems

Abstract: - In this essay, based on the Lie group analysis method, the investigation has been carried out on the symmetry properties of the (2 + 1)-dimensional Pavlov equation. Its maximal symmetry group in Lie's sense and the corresponding one-dimensional optimal system of subgroups are presented. Furthermore, by using Ibragimov's method which is a generalization of Noether's theorem, the conservation laws for the intended equation are determined. Finally, utilizing the traveling wave solutions method, some exact solutions of this nonlinear partial differential equation are constructed.

Keywords: Pavlov equation, Lie symmetry analysis, Optimal system, Solitary wave solutions, Conservation laws.

1 Introduction

The Lie group analysis method plays a crucial role in studying the properties of differential equations and obtaining exact solutions. Numerous scholars have successfully applied this method to investigate various types of nonlinear differential equations, yielding significant results. Introduced by Sophus Lie, a Norwegian mathematician in the early 19th century, the method was further developed by Ovsianikov. Its basic principle involves identifying continuous transformations with one or several parameters that preserve the equation’s invariance. The effectiveness of the Lie point symmetry approach has been widely demonstrated in nonlinear differential equations across different fields of applied science.

In their paper [6], Nardjess Benoudina, Yi Zhang, and Chaudry Masood Khalique conducted research on the Pavlov equation, which finds applications in the examination of integrable hydrodynamic chains. The Pavlov equation is given by:

$$u_{yy} + u_{xt} + u_x u_{xy} - u_{xx} = 0 \qquad (1.1)$$

The aforementioned paper provides a comprehensive overview of the equation’s history and significance. Additionally, the authors extended the Lie group method as an effective technique for solving this equation. However, the equation’s symmetry groups were not fully provided, resulting in incomplete subsequent results. Although the paper [6] has been referenced in over 20 articles, including [5, 4, 3, 2], up to April 2022, the absence of the equation’s full symmetry group limits the conclusions that can be drawn. Thus, in this essay, our aim is to rectify the computations and finalize the results.

Conservation laws, which state that the total amount of a specific physical quantity remains constant in an isolated physical system and does not change during its evolution, have diverse applications in various scientific fields. One of the most effective uses of conservation laws is in constructing singular solutions for partial differential equations, integro-differential equations, and their systems.

There exist several methods to obtain conservation laws, and in this paper, we employ Ibragimov’s method. According to Noether’s theorem, if a differential equation is an Euler Lagrange equation, its variational Lie point symmetries can be used to construct conservation laws. An Euler-Lagrange differential equation is derived from the variational principle of least action by minimizing a variational integral with a Lagrangian function as the integrand. Ibragimov introduced the concept of nonlinear self-adjointness and demonstrated that conservation laws can be obtained for differential equations that do not have Lagrangians in the classical sense [11],[12].

The structure of this paper is organized as follows: Section 2 focuses on deriving the Lie point symmetries of the Pavlov equation. In Section 3, we proceed to find a one-dimensional optimal system for the equation. Section 4 is dedicated to identifying the conservation laws associated with the equation. Furthermore, Section 5 presents some exact solutions of the Pavlov equation obtained through the traveling wave solutions method. Finally, the concluding section summarizes the findings and facilitates further discussion.

[1] *Corresponding author: Department of Pure Mathematics, School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, 16846-13114, I.R.IRAN., E-mail: m_ nadjafikhah@iust.ac.ir

[2] Author 2 : Department of financial sciences, College of Administration and Economics , University of ALQadisiyah, Iraq, P. Box 88, Email : hind.kadhum@qu.edu.iq

2 FULL SYMMETRY LIE GROUP

Among various methods, the Lie group analysis method is the most effective and yet elementary approach to constructing exact solutions of nonlinear partial differential equations. By using this method we can investigate the behavior of the Pavlov equation. Through this analysis, geometric vector fields associated with the (2 + 1)-dimensional Pavlov equation have been obtained. The Lie symmetry algebra, generated by vector fields, reveals the underlying symmetries and transformations of the equation

The general infinitesimal symmetry of Eq. (1.1) is

$$v = \tau(t,x,y,u)\partial_t + \xi(t,x,y,u)\partial_x + \eta(t,x,y,u)\partial_y + \phi(t,x,y,u)\partial_u$$

Where

$$\phi_{yyyy} = \xi_{yyy} = \eta_{yy} = \tau_y = \eta_x = \tau_x = \xi_u = \eta_u = \tau_u = 0\,, \tag{2.1}$$

$$\phi_{xx} = \phi_{xu} = \phi_{yu} = \phi_{uu} = 0\,, \tag{2.2}$$

$$\phi_y = -\xi_t\,, \phi_{tx} = -\phi_{yy}\,, \phi_{tu} = -\xi_{yy}\,, \phi_{xy} = \xi_{yy}\,, \tag{2.3}$$

$$\eta_t = -2\xi_y + \phi_x\,, \eta_y = 2\xi_x - \phi_u\,, 2\tau_t = 3\eta_y - \phi_u\,, \tag{2.4}$$

By solving the equations (2.1), it follows that:

$$\tau = \alpha(t), \xi = b(t,x)y^2 + c(t,x)y + d(t,x), \eta = e(t)y + f(t),$$
$$\phi = g(t,x,u)y^3 + h(t,x,u)y^2 + i(t,x,u)y + j(t,x,u), \tag{2.5}$$

where a, b, ….. , j are arbitrary functions. It results by placing the equations (2.5) in (2.2), (2.3) and (2.4):

$$b_x = c_x = g_x = h_x = g_u = h_u = i_u = j_{xx} = j_{xu} = j_{uu} = 0\,, \tag{2.6}$$

$$c_t + 2h = j_{tx} + 2h = j_{tu} + 2b = 2a_t - 3e + j_u = b_t + 3g = 0\,, \tag{2.7}$$

$$d_t + i = e_t + 2b = f_t + 2c - j_x = 2d_x - e - j_u = i_x - 2b = i_{tx} + 6g = 0 \tag{2.8}$$

Now by solving the equations (2.6) it follows that b = b(t) , c = c(t) , e = e(t) , f = f(t) , g = g(t) , h = h(t) , i = i(t, x) , and j = j1(t)x + j2(t)u + j3(t) , where j1, j2, j3 are arbitrary functions. By placing these expressions in (2.7) and (2.8) we have :

$$b_t + 3g = c_t + 2h = d_t + i = i_x - 2b = e_t + 2b = \acute{j}_1 + 2h = 0\,,$$
$$2d_x - e - j_2 = f_t + 2c - j_1 = 2a_t - 3e + j_2 = \acute{j}_2 + 2b = 0 \tag{2.9}$$

Solving this system of equations leads to

$$a = -2\int(\int b\,dt)\,dt + (2c_5 - c_2)t + c_6\,, c = -2\int h\,dt + c_1\,,$$
$$d = -k - 2x\int b\ dt + c_2\,x + c_3\,, e = -2\int b\ dt + c_5\,,$$
$$f = 2\int(\int h\,dt)\,dt + (c_4 - 2c_1)t + c_7\,, g = -\frac{\acute{b}}{3}\,, \tag{2.10}$$
$$i = 2bx + \acute{k}\,, j_1 = -2\int h\ dt + c_4\,, j_2 = -2\int b\ dt + 2c_2 - c_5\,,$$

where k(t) is an arbitrary function and c1,…, c7 are arbitrary constants.
Therefore,

$$\tau = f\,, \eta = \left(a + \acute{f}\right)y + g\,,$$
$$\xi = -f''\,y^2\,/\,2 + \acute{f}x + (b - \acute{g})\,y + 2\,a\,x + h\,, \tag{2.11}$$
$$\phi = f'''\,y^3/6 + g''\,y^2/2 - f''\,xy + (2b - \acute{g})x - \acute{h}y + (3a + \acute{f})u = \ell\,,$$

where f(t), g(t), h(t), ℓ(t) are arbitrary functions and a, b are arbitrary constants.
The arguments described above can be summarized as follows:

Theorem 2.1. Lie symmetry algebra g of the equation (1.1) is generated by the vector fields:

$$
\begin{aligned}
&A = 2x\,\partial x + y\partial y + 3u\partial u \ , \ B = y\partial x + 2x\partial u \ , \ C_f = f\partial u \ , \\
&D_f = f\partial x - \acute{f} y\partial u \ , \qquad E_f = -\acute{f} y\partial x + f\partial y + \left(\frac{f''}{2}\, y^2 - \acute{f} x\right)\partial u \ , \qquad (2.12) \\
&F_f = f\partial t + \left(\acute{f} x - \frac{f''}{2} y^2\right)\partial x + \acute{f} y\partial y + \left(\frac{f'''}{6}\, y^3 - f'' xy + \acute{f} u\right)\partial u\,, \ G = \partial x\,,
\end{aligned}
$$

where f(t) is an arbitrary function. The structure of this Lie algebra with non-zero Lie brackets can be introduced as follows:

$$
\begin{array}{lll}
[A,B] = -B\,, & [A,D_f] = C_{-3f}\,, & [A,D_f] = D_{-2f}\,, \\
[A,E_f] = E_{-f}\,, & [B,D_f] = C_{-2f}\,, & [B,E_f] = D_{-f}\,, \\
[C_f,F_g] = C_{f\acute{g}-\acute{f}g}\,, & [D_f,E_g] = C_{\acute{f}g-f\acute{g}}\,, & [D_f,F_g] = D_{f\acute{g}-\acute{f}g}\,, \\
[E_f,E_g] = D_{g\acute{f}-f\acute{g}}\,, & [E_h,F_\ell] = E_{h\acute{\ell}-\ell\acute{h}}\,, & [F_f,F_g] = F_{f\acute{g}=\acute{f}g}
\end{array}
$$

In other words, its Lie multiplication table is as follows:

[,]	A	B	C_f	D_g	E_h	F_ℓ	G
A	0	-B	C_{-3f}	D_{-2g}	D_{-h}	0	-2G
B	B	0	0	C_{-2g}	D_{-h}	0	$-2\partial u$
C_a	C_{3a}	0	0	0	0	$C_{a\ell-\acute{a}\ell}$	0
D_b	D_{2b}	C_{2b}	0	0	$C_{\acute{b}h-d\acute{h}}$	$D_{b\ell-\acute{b}\ell}$	0
E_c	E_c	D_c	0	$C_{\acute{b}h-d\acute{h}}$	$E_{h\acute{c}-\acute{h}c}$	$E_{c\acute{\ell}-\acute{c}\ell}$	-2C_c
F_d	0	0	$C_{\acute{f}d-f\acute{d}}$	$D_{\acute{g}d-g\acute{d}}$	$E_{h\acute{d}-\acute{h}d}$	$F_{d\acute{\ell}-\acute{d}\ell}$	$-6D_d$
G	2G	$2\partial u$	0	0	$-2C_h$	$6D_\ell$	0

Where f, g, h, ℓ, a, b, c and d are arbitrary functions of t. g is a non-commutative infinite imensional Lie algebra.

If we want all the coefficients of the generating vector fields of Lie symmetry algebra to be polynomials up to degree one, we get a 10−dimensional solvable sub-algebra g10 of g; Which is generated by the following vector fields:

$$
\begin{aligned}
&v_1 = \partial t\,, v_2 = \partial x\,, v_3 = \partial y\,, v_4 = y\partial x - 2t\partial y\,, v_5 = \partial u\,, \\
&v_6 = t\partial u\,, v_7 = t\partial x - y\partial u\,, v_8 = t\partial y + x\partial u\,, \qquad (2.13) \\
&v_9 = -y\partial y - 2t\partial t + u\partial u\,, v_{10} = x\partial x + 2y\partial y + 3t\partial t
\end{aligned}
$$

Apparently, the authors of the article [6] were looking for this issue. The Lie multiplication table of g10 is given as:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
v_1	0	0	0	$-2v_3$	0	v_5	v_2	v_3	$-2v_1$	$3v_1$
v_2		0	0	0	0	0	0	v_5	0	v_2
v_3			0	v_2	0	0	$-v_5$	0	$-v_3$	$2v_3$
v_4				0	0	0	$2v_6$	$-v_7$	v_4	$-v_4$
v_5					0	0	0	0	v_5	0
v_6						0	0	0	$3v_6$	-3v_6
v_7							0	$2v_6$	$2v_7$	$-2v_7$
v_8								0	v_8	$-v_8$
v_9									0	0
v_{10}										0

The Lie algebra considered in the paper [6] is a 9−dimensional solvable ideal g9 of g10 generated by the following vector fields:

$$v_3 , v_2 , v_1 , v_5 , 3v_9 + 2v_{10} , v_4 + 2v_8 , v_7 , -v_4 - v_8 , v_9 + v_{10} \qquad (2.14)$$

The vector field $v_6 = t\partial u$ does not belong to this sub-algebra g9, and is disregarded in [6].

If we are looking for Lie symmetry algebra with quadratic generating vector fields, we get the 14 dimensional Lie algebra g14 generated by following vector fields wi , i = 1, 2, … , 14:

$$\partial_t , \partial_x , \partial_y , y\partial_x - 2t\partial_y , \partial_u , t\partial_u , t\partial_x - y\partial_u , t\partial_y + x\partial_u ,$$

$$2x\partial_x + y\partial_y + 3u\partial_u, x\partial_x + y\partial_y + t\partial_t + u\partial_u ,$$

$$t^2\partial_u , t^2\partial_x - 2ty\partial_u , 2ty\partial_x - t^2\partial_y + (2tx - y^2)\partial_u ,$$

$$(2tx - y^2)\partial_x + 2ty\partial_y + t^2\partial_t + 2(tu - xy)\partial_u \qquad (2.15)$$

Nil radical r of this Lie algebra is generated by w1,w2, ,,, ,w11, and its semi-simple part s is generated by w12,w13,w14. Therefore, g14 decomposed as semi-direct sum $\tau \propto s$. The Lie algebra g10 is a sub-algebra of g14 generated by w1,w2,… ,w10.

If necessary, this process can be continued and Lie symmetry algebras with generators of higher degrees can be found.

3 ONE-DIMENSIONAL OPTIMAL SYSTEM OF SUBALGEBRAS

To solve a differential equation, we need to find invariant solutions that do not change with any transformation from the full symmetry group. However, there are infinitely many subgroups of the symmetry group for any given differential equation. To obtain a complete description of invariant solutions, we require subgroups that give essentially different solutions, known as the optimal system. To construct an optimal system of subgroups, we need to find an optimal system of subalgebras. For one-dimensional subalgebras, the construction of an optimal system involves the classification of the orbits for the adjoint representation. Determining the optimal system of the Pavlov equation is an essential step in understanding its behavior and solutions. By solving the equations derived from the Lie symmetry analysis, the optimal system of the Pavlov equation can be determined. This system provides valuable insights into the equation's dynamics and helps in identifying key variables and parameters.

If $v = a_1v_1 + \cdots + a_{10}v_{10}$ and $A\,d_{\exp(t_{10}v_{10})} \circ \dots \dots . Ad_{\exp(t_1v_1)}(v) = b_1v_1 + \cdots + b_{10}v_{10}$, for $a_i , b_i , t_i \in \mathbb{R}$, then

$b_1 = e^{2t_9 - 3t_{10}} \left((3a_{10} - 2a_9)t_1 + a_1\right)$,

$b_2 = e^{-t_{10}} \left(\left((2a_9 - 3a_{10})t_1 + a_1\right) t_7 + \left((2a_9 - 3a_{10})t_1 - a_1\right)t_4^2 + \left((a_9 - 2a_{10})t_3 + (2a_4 - a_8)t_1 - a_3\right)t_4 + t_2a_{10} + t_3a_4 + t_1a_7 + a_2\right)$

$b_3 = e^{t_9 - 2t_{10}} \left(\left((2a_9 - 3a_{10})t_1 - a_1\right)t_8 + (2(3a_{10} - 2a_9)t_1 + 2a_1)t_4 + (2a_{10} - a_9)t_3 + (a_8 - 2a_4)t_1 + a_3\right)$,

$b_4 = e^{t_9 - t_{10}}\left((a_9 - a_{10})t_4 + a_4\right)$,

$$b_5 = e^{-t_9}\Big(\Big(\big((3a_{10}-2a_9)t_1+a_1\big)t_7+\big((3a_{10}-2a_9)t_1+a_1\big)t_4^2+\big((2a_{10}-a_9)t_3+(a_8-2a_4)t_1+a_3\big)t_4 - t_2a_{10}-3a_4-t_1a_7-a_2\Big)t_8 + \big(2\big((3a_{10}-2a_9)t_1+a_1\big)t_4+(2a_{10}-a_9)t_3+(a_8-2a_4)t_1+a_3\big)t_7\Big),$$

$$b_6 = e^{3(t_{10}-t_9)}\Big(\big((a_{10}-a_9)t_4-a_4\big)t_8^2+2(t_4a_8+2(a_{10}-a_9)t_7-a_7)t_8+2\big((a_{10}-a_9)t_4-a_4+a_8\big)t_7 + 3(a_9-a_{10})t_6-t_4^2a_8+2t_4a_7+a_6\Big),$$

$$b_7 = e^{2(t_{10}-t_9)}\Big(\big((a_9-a_{10})t_4+a_4\big)t_8+2(a_9-a_{10})t_7-t_4a_8+a_7\Big),$$

$$b_8 = e^{t_{10}-t_9}\big((a_9-a_{10})t_8+a_8\big),$$

$$b_9 = a_9,$$

$$b_{10} = a_{10}$$

All $A\,d_{\exp(t_iv_i)}$ functions have no effect on a_9 and a_{10} The simplification algorithm of non-zero vector field v is divided into the following modes using $A\,d_{\exp(t_iv_i)}$ functions:

Case 1-0 If $a_9a_{10}(a_{10}-a_9)(2a_{10}-a_9)(3a_{10}-2a_9) \neq 0$, then by selecting the appropriate $t_1,\dots,t_8$, the vector field v can be converted to v_9+bv_{10} .

Case 1-1-1 If $a_{10}=0$, then by selecting the appropriate $t_1,\dots,t_8$, the vector field v can be converted to $\varepsilon v_3+v_4+v_9$, where $\varepsilon \in \{-1,0,1\}$.

Case 1-1-2 If$a_9=0$, then by selecting the appropriate $t_1,\dots,t_8$, the vector field v can be converted to εv_2+v_{10} , where $\varepsilon \in \{-1,0,1\}$.

Case 1-1-3-0 If $a_9=a_{10}\neq 0$ and $a_4(a_8-a_4)\neq 0$, then by selecting the appropriate $t_1,\dots,t_8$, the vector field v can be converted to $\varepsilon v_4+\delta v_8+v_9+v_{10}$, where $\varepsilon,\delta \in \{-1,0,1\}$.

Case 1-1-3-1-1-0 If $a_9=a_{10}\neq 0$, $a_4=0$ and $a_8\neq 0$, then by selecting the appropriate $t_1,\dots,t_8$, the vector field v can be converted to $v_8+v_9+v_{10}$, where $\varepsilon \epsilon \{-1,1\}$.

Case 1-1-3-1-1-1 If $a_9=a_{10}\neq 0$, $a_4=a_8=0$, then by selecting the appropriate $t_1,\dots,t_8$, the vector field v can be converted to $\varepsilon v_7+v_9+v_{10}$, where $\varepsilon \epsilon \{-1,1\}$.

Case 1-1-3-2-0 If $a_9=a_{10}\neq 0$ and $a_4=a_8\neq 0$, then by selecting the appropriate $t_1,\dots,t_8$, the vector field v can be converted to $\varepsilon(v_4+v_8)+\delta v_6+v_9+v_{10}$, where $\varepsilon \epsilon \{-1,1\}$ and $\epsilon \{-1,0,1\}$.

Case 1-1-3-2-1-0 If $a_9=a_{10}\neq 0$, $a_4=a_8=0$ and $a_8=0$, then by selecting the appropriate $t_1,\dots,t_8$, the vector field v can be converted to $\varepsilon v_7+v_9+v_{10}$, where $\varepsilon \in \{-1,0,1\}$.

Case 1-1-3-2-1-1 If $a_9=a_{10}\neq 0$, $a_4=a_8=a_7=0$, then by selecting the appropriate $t_1,\dots,t_8$, the vector field v can be converted to $\varepsilon v_6+v_9+v_{10}$, where $\varepsilon \in \{-1,0,1\}$.

Case 1-1-4 If $a_9=2a_{10}\neq 0$, then by selecting the appropriate $t_1,\dots,t_8$, the vector field v can be converted to $\varepsilon v_3+2v_9+v_{10}$, where $\varepsilon \in \{-1,0,1\}$.

Case 1-1-5 If $a_9 = (\frac{3}{2})a_{10} \neq 0$, then by selecting the appropriate $t_1, \dots, t_8$, the vector field v can be converted to $\varepsilon v_1 + 3v_9 + 2v_{10}$, where $\varepsilon \in \{-1,0,1\}$.

In general, according to the above calculations, we conclude that:

Theorem 3.1. An optimal system of one-dimensional sub-algebras of g_{10} is

$$\begin{array}{lll} av_9 + bv_{10}, & \varepsilon v_3 + v_4 + v_9, & \varepsilon v_4 + \delta v_8 + v_9 + v_{10}, \\ \varepsilon v_2 + v_{10}, & \varepsilon v_7 + v_9 + v_{10}, & \delta(v_4 + v_8) + \varepsilon v_6 + v_9 + v_{10}, \\ \varepsilon v_3 + 2v_9 + v_{10}, & \varepsilon v_1 + 3v_9 + 2v_{10}, & \end{array} \tag{3.1}$$

Where $,\delta \epsilon\{-1,0,1\}$.

4 CONSERVATION LAWS

In this section, we construct the conservation laws for the following equation:

$$F(x, y, t, u, u_x, u_{xx}, \dots) = u_{xy} + u_{xt} + u_x u_{xy} - u_y u_{xx} \tag{4.1}$$

A vector field $C = (C^t, C^x, C^y)$ is called a conserved vector for Eq. (4.1), if it satisfies the conservation equation

$$D_t(C^t) + D_x(C^x) + D_y(C^y)\Big|_{(4.1)} = 0$$

Where C^t , C^x and C^y are functions of t , x , y , u , and … [21,24] .

Equation (4.1) has a formal Lagrangian in the following form [15, 22]:

$$\mathcal{L} = \phi(t, x, y)\big(u_{yy} + u_{xt} + u_x u_{xy} - u_y u_{xx}\big) \tag{4.2}$$

Where ϕ(t, x,y) is a new dependent variable. the adjoint equation for Eq (4.1) satisfies in

$$F^* = \partial\mathcal{L}/\partial u = \lambda\big(u_{yy} + u_{xt} + u_x u_{xy} - u_y u_{xx}\big) \tag{4.3}$$

Where λ is undetermined coefficient and $\frac{\delta}{\delta u}$ is the Euler-Lagrange operator by the following definition:

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} - D_y\frac{\partial}{\partial u_y} - D_t\frac{\partial}{\partial u_t} + D_x^2\frac{\partial}{\partial u_{xx}} + D_y^2\frac{\partial}{\partial u_{yy}} + D_x D_y\frac{\partial}{\partial u_{xy}} + D_x D_t\frac{\partial}{\partial u_{xt}}$$

Therefore the component of the conserved vector are given by:

$$C^t = w\left(\frac{\partial\mathcal{L}}{\partial u_t} - D_x\frac{\partial\mathcal{L}}{\partial u_{tx}}\right) + D_x(w)\frac{\partial}{\partial u_{tx}},$$

$$C^x = w\left(\frac{\partial\mathcal{L}}{\partial u_x} - D_x\frac{\partial\mathcal{L}}{\partial u_{xx}} - D_t\frac{\partial\mathcal{L}}{\partial u_{tx}} - D_y\frac{\partial\mathcal{L}}{\partial u_{xy}}\right) + D_t(w)\frac{\partial\mathcal{L}}{\partial u_{xt}} + D_x(w)\frac{\partial\mathcal{L}}{\partial u_{xx}} + D_y(w)\frac{\partial\mathcal{L}}{\partial u_{yy}},$$

And

$$C^y = w\left(\frac{\partial\mathcal{L}}{\partial u_y} - D_x\frac{\partial\mathcal{L}}{\partial u_{xy}} - D_y\frac{\partial\mathcal{L}}{\partial u_{yy}}\right) + D_t(w)\frac{\partial\mathcal{L}}{\partial u_{yt}} + D_x(w)\frac{\partial\mathcal{L}}{\partial u_{xy}} + D_y(w)\frac{\partial\mathcal{L}}{\partial u_{yy}},$$

That $\mathscr{L}$ is the formal Lagrangian (4.2) and w defined by $= \eta - \xi^t u_t - \xi^x u_x - \xi^y u_y$.

Let us consider a Lie point symmetry generator $X = \partial_t$ with characteristic $w = -u_t$ gives the conserved vector $C = (C^1, C^2, C^3)$ such as

$$C^1 = u_t\phi_x - u_{tx}\phi \ ,$$

$$C^2 = \phi_x u_y u_t + \phi_y u_t u_x + \phi u_{xy} u_t - \phi_t u_t - \phi u_{tt} + \phi u_{tx} u_y - \phi u_{ty} \ ,$$

$$C^3 = 2\phi u_t u_{xx} + \phi_x u_x u_t + \phi_{yt} u_t - \phi u_{tx} u_x - u_{ty}\phi \ .$$

For Lie point symmetry generator $X = \partial_x$ with the characteristic $w = -u_x$ gives the conserved vector $C = (C^1, C^2, C^3)$ such as:

$$C^1 = u_x\phi_x - u_{xx}\phi \ ,$$

$$C^2 = -\phi u_x u_{xy} - \phi_x u_y u_x + \phi_y u_x^2 + \phi_t u_x - \phi u_{xt} + \phi u_{xx} u_y - \phi u_{xy} \ ,$$

$$C^3 = 2\phi u_x u_{xx} + \phi_x u_x^2 + \phi_y u_x - \phi u_{xy} \ .$$

For Lie point symmetry generator $X = \partial_y$ with the characteristic $w = -u_y$ gives the conserved vector $C = (C^1, C^2, C^3)$ such as:

$$C^1 = u_y\phi_x - u_{yx}\phi \ ,$$

$$C^2 = -\phi_x u_y^2 + \phi_y u_x u_y - \phi_t - \phi u_{yt} - \phi u_{yy} \ ,$$

$$C^3 = 2\phi u_y u_{xx} + \phi_x u_x u_y - \phi u_{xy} u_x - \phi_y - \phi u_{yy} \ .$$

For Lie point symmetry generator $X = -y\partial x + 2t\partial y$ with the characteristic $w = -yu_x + 2tu_y$ gives the conserved vector $C = (C^1, C^2, C^3)$ such as:

$$C^1 = tyu_x\phi_x + 2tu_y\phi_x - yu_{xx}\phi + 2tu_{yx}\phi \ ,$$

$$C^2 = -yu_x u_{xy}\phi + 2tu_y^2\phi_x + yu_x^2\phi_y - 2tu_y u_x\phi_y - \phi_t - yu_{xt}\phi + 2u_y\phi + 2tu_{yt}\phi + yu_{xx}u_y\phi - u_x - yu_{xy}\phi_t + 2tu_{yy} \ ,$$

$$C^3 = -4tu_y u_{xx}\phi + yu_x^2\phi_x - 2tu_y u_{xx}\phi_{xx} + yu_x u_{xx}\phi + 2tu_{xy}u_x\phi - u_x\phi - yu_{xy}\phi + 2tu_{yy}\phi \ .$$

For Lie point symmetry generator $X = t\partial x - y\partial u$ with the characteristic $w = -tu_x - y$ gives the conserved vector $C = (C^1, C^2, C^3)$ such as:

$$C^1 = tu_x\phi_x + y\phi_x - tu_{xx}\phi \ ,$$

$$C^2 = -tu_x u_y\phi_x - tu_x u_{xy}\phi + tu_x^2\phi_y - u_x\phi - tu_{xt}\phi + tu_{xx}u_y\phi + tu_{xy}\phi - \phi - tu_x\phi_t + y\phi_t \ ,$$

$$C^3 = tu_x^2\phi_x + tu_x u_{xx}\phi + tu_x\phi_y + 2yu_{xx}\phi + yu_x\phi_x + y\phi_y - tu_{xy}\phi - \phi \ .$$

For Lie point symmetry generator $X = t\partial y + x\partial u$ with the characteristic $w = x - tu_y$ gives the conserved vector $C = (C^1, C^2, C^3)$ such as:

$$C^1 = -x\phi_x + tu_y\phi_x + \phi - tu_{xy}\phi \ ,$$

$$C^2 = x\phi u_{xy} + x\phi_x u_y - x\phi_y u_x - x\phi_t - u_y\phi - tu_{yt}\phi - \phi u_y + tu_{xy}u_y\phi - tu_{yy}\phi \ ,$$

$$C^3 = 2x\phi u_{xx} - x\phi_x u_{xx} - x\phi_y + 2u_y u_{xx}\phi + tu_y u_x \phi_x + tu_y \phi_y + \phi u_x - tu_y u_x \phi - tu_{yy}\phi \,.$$

For Lie point symmetry generator $X = u\partial u - y\partial y - 2t\partial t$ with the characteristic $w = u + yu_y + 2tu_t$ gives the conserved vector $C = (C^1, C^2, C^3)$ such as:

$$C^1 = -u\phi_x - yu_y\phi_x - 2tu_t\phi_x + u_x\phi + yu_{xy}\phi + 2tu_{tx}\phi \,,$$

$$\begin{aligned} C^2 = {} & uu_y\phi_x + uu_{xy}\phi - uu_x - \phi_y - u\phi_t + u_t\phi + yu_{ty}\phi + 2tu_t\phi + 2tu_{tt}\phi - u_x u_y\phi \\ & -u_{xy}u_y\phi - 2tu_t u_y\phi + 2u_y\phi + yu_{yy}\phi + 2tu_{ty}\phi, \\ C^3 = {} & -\phi uu_{xx} - 2yu_y u_{xx}\phi - \phi_x u_x u - \phi_y u - \phi_x yu_y u_x - \phi_y yu_y - 4tu_t u_{xx}\phi - 2tu_t u_x\phi_x \\ & -2tu_t\phi_y + u_x^2\phi + yu_{xy}u_x\phi + 2tu_{tx}u_x\phi + 2u_y\phi + yu_{yy}\phi + 2tu_{ty}\phi \,. \end{aligned}$$

For Lie point symmetry generator $X = x\partial x + 2y\partial t + 3t\partial t$ with the characteristic $w = -xu_x - 2yu_y - 3tu_t$ gives the conserved vector $C = (C^1, C^2, C^3)$ such as:

$$C^1 = -xu_x\phi_x + 2yu_y\phi_x + 3tu_t\phi_x - u_x\phi - xu_{xx}\phi - 2yu_{xy}\phi - 3tu_{tx}\phi \,,$$

$$\begin{aligned} C^2 = {} & -xu_x u_y\phi_x - xu_x u_{xy}\phi + xu_x^2\phi_y - xu_x\phi_t - 2yu_y^2\phi + 2yu_y u_x\phi_x + 2yu_y\phi_t - 3tu_t u_{xy}\phi \\ & -3tu_t u_y\phi_x + 3tu_t u_x\phi_y + 3tu_t\phi_t - xu_{xt}\phi - 2yu_{yt}\phi - 3u_t\phi - 3tu_{tt}\phi + u_x u_y\phi + \\ & xu_{xx}u_y\phi + 3tu_{tx}u_y\phi - xu_{xy}\phi - 2u_y\phi - 2yu_{yy}\phi - 3tu_{ty}\phi, \end{aligned}$$

$$\begin{aligned} C^3 = {} & xu_x u_{xx}\phi + xu_x^2\phi_x + xu_x\phi_y + 4yu_y u_x x\phi + 2yu_y u_x\phi_x + 2yu_y\phi_y + 6tu_t u_{xx}\phi \\ & +3tu_t u_x\phi_x + 3tu_t\phi_y - u_x^2\phi - 2yu_{xy}u_x\phi - 3tu_{tx}u_x\phi - xu_{xy}\phi - 2u_y\phi - 2yu_{yy}\phi - 3tu_{ty}\phi \,. \end{aligned}$$

Where $\phi = c_1txy + c_2tx + c_3ty + c_4xy + c_5t + c_6x + c_7y + c_8$.

5 TRAVELING WAVE SOLUTIONS

The tanh technique has been employed to generate accurate traveling wave solutions for the Pavlov equation. By introducing a new variable and using

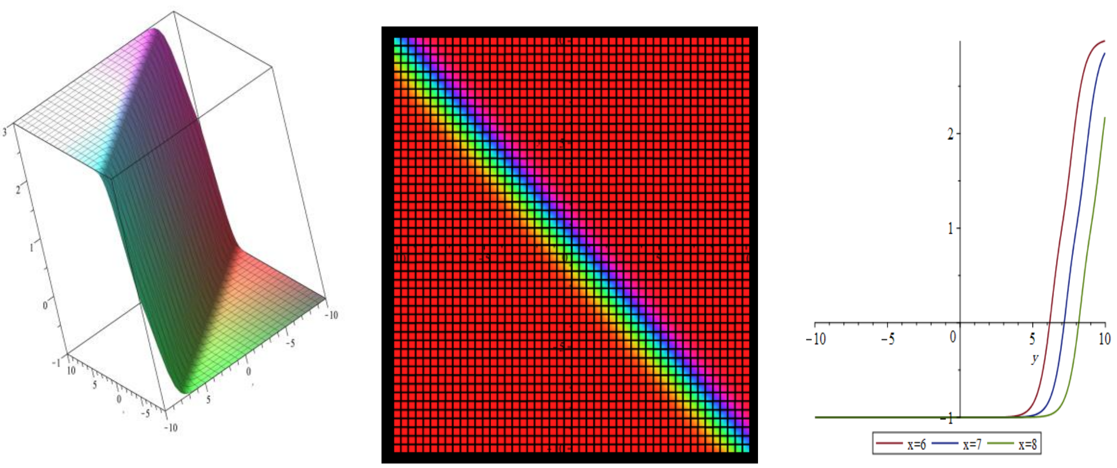

Figure 1: The graph of the u_2 , $c_i = 1$, $t = 1$.

the Ansatz method, exact solutions can be obtained. These new solitary wave solutions offer valuable insights into the behavior and characteristics of the Pavlov equation, providing a deeper understanding of its dynamics.
In this section, we perform one of the most important ansatz methods (the tanh-function method, [1]) to gain exact traveling wave solutions of this nonlinear system of PDEs. For this, we introduce a new variable $= \tanh(c_0 + c_1 t + c_2 x + c_3 y)$, where c_i are arbitrary constants. By placing this expression in the equation (2.1), we get

$$(c_2^2(\tau^2-1)^2 + c_3(\tau^2-1)^2 c_1)u^2 + (2c_2^2(\tau^2-1)\tau + 2c_3(\tau^2-1)c_1\tau)u = 0$$

Then, using the ansatz $= A_{1,0} + A_{1,1}\tau + A_{1,2}\tau^2 + A_{1,3}\tau^3$, where $A_{1,i}$ are arbitrary constants, we obtain the exact solution by using required simplifications and linear algebra:

1) $u_1 = a_1$,
2) $u_2 = c_1 tanh^3(\zeta) + c_5 \tanh(\zeta) + c_6$,
3) $u_3 = c_1 tanh^3(\zeta) + c_7 tanh^2(\zeta) + c_5 \tanh(\zeta) + c_6$.

Where $\zeta = -\frac{c_2^2 t}{c_3} + c_3 x + c_2 y + c_4$. In figure 1 and figure 2, solutions have been plotted for some constant coefficients.

6 CONCLUSION

In this article, we successfully presented a systematic technique for obtaining Lie point symmetries of the Pavlov equation. Additionally, a simple method was utilized to construct the optimal system of the full Lie algebras of the equation. This allowed for the classification of subgroups, enabling the construction of all inequivalent reduced ordinary differential equations (ODEs) and the derivation of explicit exact solutions.

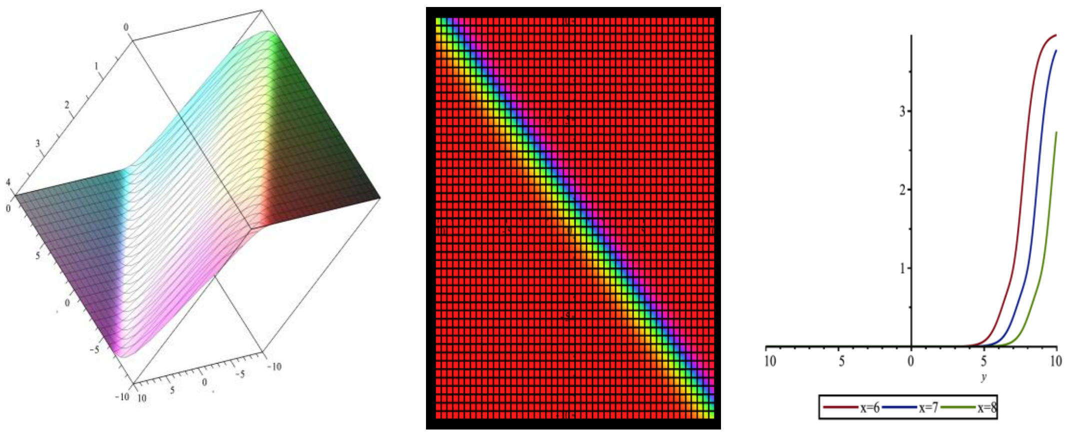

Figure 2: The graph of the u_2 , $c_i = 1$, $t = 1$.

The derived Lie point symmetries were employed to derive conservation laws for the Pavlov equation using Ibragimov's method. Furthermore, the tanh technique was utilized to generate accurate traveling wave solutions for this nonlinear partial differential equation.

REFERENCES

[1] Malfliet, Willy and Hereman, Willy, The tanh method: "I. Exact solutions of nonlinear evolution and wave equations," IOP Publishing, Physica Scripta. 54 (6) (1996) 661–680.

[2] Li, Xinyue and Zhang, Yongli and Zhang, Huiqun and Zhao, Qiulan, "Lie symmetry analysis and conservation laws for the (2+1)-dimensional Mikhalev equation," IOP Publishing, Physica Scripta. (2021) Texas State University, Department of Mathematics.

[3] Kumar, Sachin and Kumar, Mukesh and Kumar, Dharmendra, "Computational soliton solutions to (2+ 1)-dimensional Pavlov equation using Lie symmetry approach," Springer, Pramana. 94 (1) (2020) 28.

[4] Khalique, Chaudry Masood, "Closed-form solutions and conservation laws of a generalized Hirota–Satsuma coupled KdV system of fluid mechanics," De Gruyter, Open Physics. 19 (1) (2021) 18-25.

[5] Adeyemo, Oke Davies and Khalique, Chaudry Masood, "Analytic solutions and conservation laws of a (2+ 1)-dimensional generalized Yu–Toda–Sasa–Fukuyama equation," Elsevier, Chinese Journal of Physics. 77 (2022) 927-944.

[6] Benoudina, Nardjess and Zhang, Yi and Khalique, Chaudry Masood, "Lie symmetry analysis, optimal system, new solitary wave solutions and conservation laws of the Pavlov equation, "Elsevier, Communications in Nonlinear Science and Numerical Simulation. 94 (2021).

[7] Oldham, Keith B, and Spanier, Jerome, "The Fractional Calculus," vol. 111 of Mathematics in science and engineering, Academic Press, New York, London (1974).

[8] Kilbas, AA and Marichev, OI and Samko, SG," Fractional integral and derivatives (theory and applications)," Gordon and Breach, Switzerland (1993).

[9] Gazizov, RK and Kasatkin, AA and Lukashchuk, S Yu, "Continuous transformation groups of fractional differential equations," Vestnik Usatu. 9 (3) (2007) 21 pages.

[10] Atanackovi c, Teodor M and Konjik, Sanja and Pilipovi c, Stevan and Simi c, Srboljub, "Variational problems with fractional derivatives: invariance conditions and Noether's theorem, Elsevier, Nonlinear Analysis: Theory, Methods & Applications." 71 (5) (2009) 1504- 1517.

[11] Ibragimov, Nail H, "A new conservation theorem, Elsevier," Journal of Mathematical Analysis and Applications. 333 (1) (2007) 311-328.

[12] Ibragimov, Nail H, "Nonlinear self-adjointness and conservation laws," IOP Publishing, Journal of Physics A: Mathematical and Theoretical. 44 (43) (2011).

[13] Noether, Emmy,"Invariant variation problems Transport Theory Stat,"Phys. 1 (1971) 186-207.

[14] Gandarias, ML, "Conservations laws for a porous medium equation through nonclassical generators," Elsevier, Communications in Nonlinear Science and Numerical Simulation. 19 (2) (2014) 371-376.

[15] Lukashchuk, Stanislav Yu, "Conservation laws for time-fractional subdiffusion and diffusion-wave equations," Springer, Nonlinear Dynamics. 80 (1-2) (2014) 791-802.

[16] Ismail, Hassan NA and Raslan, Kamal and Rabboh, Aziza A Abd, "Adomian decomposition method for Burger's-Huxley and Burger's-Fisher equations," Elsevier, Applied Mathematics and Computation. 159 (2) (2004) 291-301.

[17] Mickens, RE and Gumel, Abba, "Construction and analysis of a non-standard finite difference scheme for the Burgers-Fisher equation" Academic Press Inc., Journal of sound and vibration. 257 (4) (2002) 791-797.

[18] Miller, Kenneth S and Ross, Bertram, "An introduction to the fractional calculus and fractional differential equations," Wiley. (1993).

[19] Yasar, Emrullah, "On the Lie symmetry analysis and traveling wave solutions of time fractional fifth-order modified Sawada-Kotera equation," Karaelmas Science and Engineering Journal. 8 (2) (2002) 411-416.

[20] Kiryakova, Virginia S, "Generalized fractional calculus and applications," CRC press. (1993).

[21] Odzijewicz, Tatiana and Malinowska, "Agnieszka and Torres, Delfim, Noether's theorem for fractional variational problems of variable order," Versita, Open Physics. 11(6) (2013) 691-701.

[22] Gazizov, RK and Ibragimov, NH and Lukashchuk, S Yu, "Nonlinear self-adjointness, conservation laws and exact solutions of timefractional Kompaneets equations," Elsevier,Communications in Nonlinear Science and Numerical Simulation. 23(1-3) (2015) 153-163.

[23] Kilbas, A Anatolii Aleksandrovich and Srivastava, Hari Mohan and Trujillo, Juan J, "Theory and applications of fractional differential equations," Elsevier. 204 (206).

[24] Golbabai, A and Javidi, M, "A spectral domain decomposition approach for the generalized Burger's–Fisher equation," Elsevier,Chaos, Solitons & Fractals. 39(1) (2009) 385–392.

[25] Chandraker, Vinay and Awasthi, Ashish and Jayaraj, Simon, "Numerical Treatment of Burger-Fisher equation," Elsevier,Procedia Technology. 25 (2016) 1217-1225.

[26] Lie, Sophus, "Theorie der Transformationsgruppen Abschn." 3, Teubner (Leipzig) (1893) 1217-1225.

[27] Babolian, E and Saeidian, J, "Analytic approximate solutions to Burgers, Fisher, Huxley equations and two combined forms of these equations," Elsevier,Communications in Nonlinear Science and Numerical Simulation. 14(5) (2009) 1984-1992.

[28] Lu, Jiang and Yu-Cui, Guo and Shu-Jiang, Xu, "Some new exact solutions to the Burgers–Fisher equation and generalized Burgers– Fisher equation," IOP Publishing,Chinese Physics. 16(9) (2007) 2514.

[29] Cai, Guoliang and Wang, Yan and Zhang, "Fengyun, Nonclassical symmetries and group invariant solutions of Burgers-Fisher equations," Elsevier,World Journal of Modelling and Simulation. 3(4) (2007) 305-309.

[30] Rosa, M and Camacho, J Carlos and Bruz´on, MS and Gandarias, Maria Luz, "Conservation laws, symmetries, and exact solutions of the classical Burgers-Fisher equation in two dimensions," Elsevier, Journal of Computational and Applied Mathematics. 354 (2019) 545-550.

[31] Rosa, M and Camacho, J Carlos and Bruz´o, "An analytic study of Fisher's equation by using Adomian decomposition method," Elsevier, Applied Mathematics and Computation. 154(3) (2004) 609–620.

[32] Wazwaz, Abdul-Majid , "The tanh method for generalized forms of nonlinear heat conduction and Burgers-Fisher equations," Elsevier, Applied Mathematics and Computation. 169(1) (2005) 321-338.