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The Primitive Exponent Set of a Class of Representative Two-colored Digraph in Graph Theory



Abstract: - Graph is simple and intuitive. It can be used to solve many problems in computer science. For a kind of representative digraph, the edges (arcs) of the digraph are colored with red and blue colors. The range of the primitive exponent are discussed in different cases, and the extremal two-colored digraphs are found by coloring all arcs with two colors. Finally, the primitive exponent set is given. The results can provide a reference for the study of primitive exponent of three-colored digraph and the application of graph coloring in computer science, such as communication network and coding cache.

Keywords: Two-colored, Digraph, Primitive Exponent, Set, Computer Science, Application.

I. INTRODUCTION

Graph theory is an important branch of combinatorial mathematics, which has the characteristics of turning complex into simple, flexible, and intuitive, and originated from the study of Königsberg's Seven Bridges Problem. With a large number of problems related to graph theory, such as four-color problem, Mitsui and three houses problem, traveling around the world, etc., graph theory has been widely used in various fields such as physics, chemistry, computer science, communication network, control engineering, social science and management science. For example, the concept of tree in graph theory can be used to solve problems in circuit theory. In the chemical molecular structure, each atom in the molecule can be regarded as each vertex in the graph, and the chemical bond between atoms can be regarded as each edge in the graph, which can solve the problem of molecular structure in chemistry. In the communication network, we can use graph theory to solve the problem of communication network by representing the communication station as the vertex in the graph and the communication line between the communication station as the edge in the graph.

At present, with the rapid development of computer, graph theory has been widely used in the field of computer, not only in the design of communication network and switching network, but also in the combinatorial optimization calculation of neural network and coding theory. Some specific applications of graph theory in computer are given and some results are obtained [1-3]. The research content of graph theory is very extensive, such as vertex and edge coloring problem, vertex covering problem, maximal clique and maximal independent set of graph and so on. In fact, many problems in computer-related fields can be transformed into graph theory problems. In this paper, the corresponding relationship between graph and matrix is used to solve the coloring problem of a class of graph edges in communication network. The following is the relevant basic knowledge required in this paper.

Let D be a digraph and $V = \{v_1, v_2, \dots, v_{d+1}\}$ be the set of vertices. $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_{d+1}$ is a walk of length d from v_1 to v_{d+1} in D . If the vertices in a walk are different, it is called a *path*. If the beginning and ending of a walk are coincided and the vertices in the walk are different, the walk is called a *cycle*. A *two-colored digraph* D is one in which the arcs in the digraph are colored by any two colors, and let's assume that the arcs in D are colored by red and yellow. If ϖ is a walk in D , then ϖ can be decomposed into the vector $(r(\varpi), y(\varpi))$ or $(r(\varpi), y(\varpi))^T$, where $r(\varpi)$ and $y(\varpi)$ represent the number of red and yellow arcs in ϖ . If any two vertices in D have a walk to get there, then D is *strongly connected* [1].

The necessary and sufficient condition for two-colored digraph D to be *primitive* is that there exists an (h_1, h_2) -walk for every pair of vertices. The minimum value of $h_1 + h_2$ is called the *primitive exponent* of D , usually written as $\exp(D)$, where h_1, h_2 are nonnegative integers and $h_1 + h_2 > 0$ [1].

If D contains d cycles, $C = \{C_1, C_2, \dots, C_d\}$ is the set of cycles, and the corresponding cycle matrix

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$$M_{2 \times d} = \begin{pmatrix} r_1 & r_2 & \cdots & r_d \\ y_1 & y_2 & \cdots & y_d \end{pmatrix}$$

for some nonnegative integers $r_i, y_i (i = 1, 2, \dots, d)$. r_i and y_i represent the number of red and yellow arcs in the C_i -cycle, respectively. The content of $M_{2 \times d}$, usually written as $\text{content}(M_{2 \times d})$, represents the capacity of $M_{2 \times d}$. The $\text{content}(M_{2 \times d}) = 0$ if the rank of $M_{2 \times d}$ is less than 2, or else $\text{content}(M_{2 \times d})$ is equal to the greatest common factor of all the second-order subformulas of $M_{2 \times d}$.

Lemma 1.1[2] The necessary and sufficient condition for two-colored digraph D to be primitive is that $\text{content}(M_{2 \times d}) = 1$ and D is strongly connected.

A closed digraph with n vertices can be associated with a nonnegative matrix of order n . With the improvement of the study of the primitive exponents of a single matrix, it is an inevitable trend to extend a single matrix to a pair of matrices. Similarly, the corresponding relationship between a pair of matrices and a bicolor digraph can be established. For each pair of nonnegative matrix (A, B) , in which $A = a_{ij}, B = b_{ij}$, it is accompanied by the seal for $D(A, B)$. According to the correspondence, we can know that the value of the element in the matrix a_{ij} is non-zero or zero corresponds to whether there is a red arc in $D(A, B)$. If the element $a_{ij} > 0$ (or $a_{ij} = 0$), then there is a (or no) red arc from vertex i to vertex j ; The values of elements in the matrix b_{ij} are non-zero or zero corresponds to whether there is a blue arc in $D(A, B)$, and if the element $b_{ij} > 0$ (or $b_{ij} = 0$), then there is a (or no) blue arc from vertex i to vertex j .

Similar to the correspondence between a single nonnegative matrix and its associated digraph, the nonnegative matrix pairs also corresponds to their associated digraph. And because digraphs are usually more intuitive than matrices, we usually solve the primitive exponent problems of nonnegative matrix pairs by corresponding two-colored digraph. At present, some achievements of two-colored digraphs have been made, but the researches are generally aimed at some special two-colored digraphs, and the upper bound, lower bound and extremal graphs characterization of primitive exponents have been found. The research on the primitive exponent set of two-colored digraph is limited to some special cases and is not representative to a certain extent.

Some basic definitions for the development of this paper are given in [4]. The correspondence between the nonnegative matrix pair and a digraph is given, and the important conclusion of Lemma 1.1 is obtained in [5]. In [6-7], the problems of two kinds upper bound of primitive index of bicolor digraph have studied and corresponding results are obtained. In [8-10], some scholars have systematically studied the exponential problem of bicolor graphs, but have hardly mentioned the exponential set problem. With the in-depth study of bicolor digraph, some experts extend the primitive index problem to the scrambling index and competition index problem, and get some results [11-17]. With the deepening of the research, most of the problems about the original index of bicolor directed graph have been solved relatively well, but the problem of index set is still a difficult problem. In [18-19], two simple bicolor digraphs are studied, and the primitive exponent sets are obtained, but the selected digraphs are special and limited. In [20-21], the authors extend the method of bicolor digraph to trichromatic digraph, and find the upper bounds of some special trichromatic digraphs.

In this paper, we select a representative digraph containing two cycles, which have unfixed common arcs mainly reflected in the two cycles, and can intersect at one point, or one common arc, or two common arcs, or even more common arcs, that is, including all the possibilities for the intersection of two cycles. For some nonnegative integers m, p, q and $m \geq 1, p \leq m$. A representative two-colored digraph is studied, and its uncolored condition is shown in Figure 1.

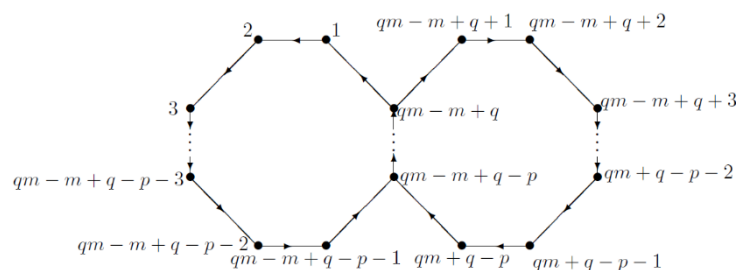


Figure 1: Uncolored Digraph of D

From Figure 1, we know D is strongly connected and contains only one $(qm-m+q)$ -cycle and one $(m+1)$ -cycle. The two cycles have p common arcs, that is, $qm-m+q-p \rightarrow qm-m+q-p+1 \rightarrow \dots \rightarrow qm-m+q$ are common arcs. We can suppose

$$M_{2 \times 2} = \begin{pmatrix} a_1 & a_2 \\ qm-m+q-a_1 & m+1-a_2 \end{pmatrix} \tag{1}$$

for some nonnegative integers $a_1, qm-m+q-a_1, a_2, m+1-a_2$. For the convenience of expression, in the following chapters, we uniformly stipulate that $M_{2 \times 2}$ is represented by M . In addition, due to the high computational complexity of the article, in order to ensure the correctness of the calculation, the following chapters all use ‘maple’ for calculations.

II. THE PRIMITIVE CONDITIONS

According to Formula (1), we know every element is a nonnegative integer in the cycle matrix M , so $0 \leq a_1 \leq qm-m+q, 0 \leq a_2 \leq m+1$. In this section, we will give the primitive conditions for D by case.

Theorem 2.1 If D is primitive, then $a_1 = q-1, a_2 = 1$ or $a_1 = qm-m, a_2 = m$.

Proof Formula (1), $|M| = a_1(m+1) - a_2(qm-m+q)$. By Lemma 1.1, we know the necessary and sufficient condition for D to be primitive is that $\text{content}(M) = 1$, so

$$|M| = a_1(m+1) - a_2(qm-m+q) = \pm 1. \tag{2}$$

According to Formula (2), we can talk about it in two ways.

Case 1: $|M| = -1$.

At this time, $a_1 = \frac{a_2(qm-m+q)-1}{m+1} = a_2(q-1) + \frac{a_2-1}{m+1}$. Since a_1 is a nonnegative integer, the value of a_1 depends on $\frac{a_2-1}{m+1}$. Because of $0 \leq a_2 \leq m+1$, so $\frac{a_2-1}{m+1} = 0$, that is $a_1 = q-1, a_2 = 1$.

Case 2: $|M| = 1$.

At this time, $a_1 = \frac{a_2(qm-m+q)+1}{m+1} = a_2(q-1) + \frac{a_2+1}{m+1}$. Since a_1 is a nonnegative integer, the value of a_1 depends on $\frac{a_2+1}{m+1}$. Because of $0 \leq a_2 \leq m+1$, so $\frac{a_2+1}{m+1} = 1$, that is $a_1 = qm-m, a_2 = m$.

The values of a_1, a_2 are respectively substituted into Formula (1), we can see that if $|M| = -1$, then

$$M_{2 \times 2} = \begin{pmatrix} q-1 & 1 \\ qm-m+1 & m \end{pmatrix} \tag{3}$$

and if $|M| = 1$, then

$$M_{2 \times 2} = \begin{pmatrix} qm-m+1 & m \\ q-1 & 1 \end{pmatrix}.$$

In both cases, the numbers of red arc are exactly reversed with the number of yellow arcs in M , so we will only discuss the case of $|M| = -1$ below, and

$$(M_{2 \times 2})^{-1} = \begin{pmatrix} -m & 1 \\ qm-m+1 & -q+1 \end{pmatrix}. \tag{4}$$

III. THE PRIMITIVE EXPONENTIAL RANGES AND EXTREMAL TWO-COLORED DIGRAPHS

Since q is a nonnegative integer, we can find if $q=0$, then $q-1 < 0$ in Formula (3). Obviously, this is not reasonable, so we will only discuss the case where $q \geq 1$. In this chapter, we will give the primitive exponential bounds in different cases, and describe the extremal two-colored digraphs which reach the upper and lower bounds of exponents. For ease of expression, let's assume that p_{v_i, v_j} is the shortest path between v_i and v_j for any vertices (v_i, v_j) in D , denoted as $r(p_{v_i, v_j}) = r$ and $y(p_{v_i, v_j}) = y$. We assume that the walk starts at vertex v_i and follow

p_{v_i, v_j} to vertex v_j by going around $(qm-m+q)$ -cycle p_1 times and $(m+1)$ -cycle p_2 times, is decomposed. When p_1 and p_2 taking different values, the corresponding walk decomposition will also change, thus obtaining the corresponding value of $\exp(D)$. In addition, let X represent a column matrix with two rows and one column. The elements in X are greater than or equal to 0 and are integers.

A. *The primitive exponent and the extremal two-colored digraph for $q=1$*

Theorem 3.1 If $q=1$, D is primitive, $\exp(D)=2m+1$ if and only if all yellow arcs must be consecutive in D .

Proof The values of a_1, a_2 are respectively substituted into Formulas (3) and (4), we can get the length of the two cycles is 1 and $m+1$ respectively, that is, the $(qm-m+q)$ -cycle is a loop and there is only one red arc and m yellow arcs on the $(m+1)$ -cycle. Obviously, the m yellow arcs must be continuous on the $(m+1)$ -cycle. The following is discussed in two ways.

Case 1: $\exp(D) \geq 2m+1$.

First, we can take v_i and v_j as the starting and ending vertices of the m consecutive yellow arcs on the $(m+1)$ -cycle, then it is decomposed into $(0, m)$. So

$$MX = \begin{pmatrix} h_1 \\ h_2 - m \end{pmatrix}$$

has a solution. Therefore,

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} 0 \\ m \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} m \\ 0 \end{pmatrix}.$$

Then $u_1 \geq m$. Next, we take v_i and v_j as the ending and starting vertices of the m consecutive yellow arcs on the $(m+1)$ -cycle, then it is decomposed into $(1, 0)$. So

$$MX = \begin{pmatrix} h_1 - 1 \\ h_2 \end{pmatrix}$$

has a solution. Therefore,

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} -m \\ 1 \end{pmatrix}.$$

Then $u_2 \geq 1$. Thus

$$h_1 + h_2 = (1 \ 1)M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^T \geq (1 \ m+1)(m \ 1)^T = 2m+1.$$

That is $\exp(D) = h_1 + h_2 \geq 2m+1$.

Case 2: $\exp(D) \leq 2m+1$.

Combined with the corresponding cycle matrix, we have $0 \leq r \leq 1, 0 \leq y \leq m$. Taking $p_1 = m+mr-y$ and $p_2 = 1-r$, we see that

$$\begin{pmatrix} r \\ y \end{pmatrix} + p_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + p_2 \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} m+1 \\ m \end{pmatrix}.$$

Obviously, $p_1 \geq 0$ and $p_2 \geq 0$. This gives $\exp(D) \leq m+1+m = 2m+1$.

B. *Exponential upper bound and the extremal two-colored digraphs for $q > 1, p=0$*

Theorem 3.2 If $q > 1, p=0, D$ is primitive, then $\exp(D) = 2q^2m^2 - 2qm^2 + 2q^2m + qm - m + 2q$ if and only if there exists a continuous yellow path of $qm+1$ length or a continuous red path of q length in D .

Proof When $q > 1, p=0$, the $(qm-m+q)$ -cycle and the $(m+1)$ -cycle intersect at the point $qm-m+q$.

At this point, the two cycles have q red arcs and $qm+1$ yellow arcs in D . Combine Formulas (3) and (4), the discussion is divided into the following three situations.

Case 1: $\exp(D) \leq 2q^2m^2 - 2qm^2 + 2q^2m + qm - m + 2q$.

Taking $p_1 = qm+1+mr-y$ and $p_2 = q^2m - qm + q - (qm - m + 1)r + (q-1)y$, we can get that

$$\begin{pmatrix} r \\ y \end{pmatrix} + p_1 \begin{pmatrix} q-1 \\ qm-m+1 \end{pmatrix} + p_2 \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} (qm+1)(q-1) + q^2m - qm + q \\ (qm+1)(qm-m+1) + q^2m^2 - qm^2 + qm \end{pmatrix}.$$

Noting that $0 \leq r \leq q$ and $0 \leq y \leq qm+1$. If $r = q$, then $y \geq 0$. If $y = qm+1$, then $r \geq 0$. Obviously, $p_1 \geq 0$ and $p_2 \geq 0$. This gives

$$\begin{aligned} \exp(D) &\leq (qm+1)(q-1) + q^2m - qm + q + (qm+1)(qm-m+1) + q^2m^2 - qm^2 + qm \\ &= 2q^2m^2 - 2qm^2 + 2q^2m + qm - m + 2q. \end{aligned}$$

Case 2: $\exp(D) \geq 2q^2m^2 - 2qm^2 + 2q^2m + qm - m + 2q$.

At this time, there exists a continuous yellow path of $qm+1$ length or a continuous red path of q length in D . Without loss of generality, we can take v_i and v_j as the starting and ending vertices of the $qm+1$ consecutive yellow arcs, or the ending and starting vertices of the q consecutive red path, then it is decomposed into $(0, qm+1)$. So

$$MX = \begin{pmatrix} h_1 \\ h_2 - (qm+1) \end{pmatrix}$$

has a solution. Therefore,

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} 0 \\ qm+1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} qm+1 \\ -(q-1)(qm+1) \end{pmatrix}.$$

Then $u_1 \geq qm+1$. Next, we take v_i and v_j as the ending and starting vertices of the $qm+1$ consecutive yellow path, or the starting and ending vertices of the q consecutive red path, then it is decomposed into $(q, 0)$. So

$$MX = \begin{pmatrix} h_1 - q \\ h_2 \end{pmatrix}$$

has a solution. Therefore,

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} q \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} -qm \\ q(qm-m+1) \end{pmatrix}.$$

Then $u_2 \geq q(qm-m+1)$. Thus

$$h_1 + h_2 = (1 \ 1)M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \geq (qm-m+q \ m+1)(qm+1 \ q(qm-m+1))^T = 2q^2m^2 - 2qm^2 + 2q^2m + qm - m + 2q.$$

That is $\exp(D) \geq 2q^2m^2 - 2qm^2 + 2q^2m + qm - m + 2q$.

Case 3: If the $qm+1$ yellow arcs are not consecutive then $\exp(D) < 2q^2m^2 - 2qm^2 + 2q^2m + qm - m + 2q$.

In this case, there are at most qm long continuous yellow path and $q-1$ long continuous red path in D .

Taking $p_1 = qm+mr-y$ and $p_2 = q^2m - qm+1 - (qm-m+1)r + (q-1)y$, we see that

$$\begin{pmatrix} r \\ y \end{pmatrix} + p_1 \begin{pmatrix} q-1 \\ qm-m+1 \end{pmatrix} + p_2 \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} qm(q-1) + q^2m - qm + 1 \\ qm(qm-m+1) + q^2m^2 - qm^2 + m \end{pmatrix}.$$

Noting that $0 \leq r \leq q$ and $0 \leq y \leq qm+1$. If $r = q$, then $y \geq 1$. If $r = q-1$, then $y \geq 0$. If $y = qm+1$, then $r \geq 1$. If $y = qm$, then $r \geq 0$. Obviously, $p_1 \geq 0$ and $p_2 \geq 0$. This gives

$$\begin{aligned} \exp(D) &\leq qm(q-1) + q^2m - qm + 1 + qm(qm-m+1) + q^2m^2 - qm^2 + m \\ &= 2q^2m^2 - 2qm^2 + 2q^2m - qm + m + 1 < 2q^2m^2 - 2qm^2 + 2q^2m + qm - m + 2q. \end{aligned}$$

To sum up, then the theorem follows.

C. Exponential upper bound and the extremal two-colored digraphs for $q > 1, p \geq 1$

Form Figure.1, we can see $0 \leq p \leq m$. When $p \geq 1$, the $(qm-m+q)$ -cycle and the $(m+1)$ -cycle have one common arc at least. Since there is only one red arc on the $(m+1)$ -cycle, this red arc may be on the common arcs. Therefore, the upper bounds of the primitive exponent will be discussed in two cases, and the extremal graph description that reaches the upper bound of exponents will be given.

Theorem 3.3 If $q > 1, p \geq 1$, the common arcs contain a red arc and D is primitive, then

$$\exp(D) = 2q^2m^2 - 4qm^2 + 2q^2m + 2m^2 - qm - m + 2q - 1$$

if and only if there exists a continuous red path of $q-1$ length in D .

Proof Similar proof of theorem 3.2, it can be proved in the following three cases.

Case 1: $\exp(D) \leq 2q^2m^2 - 4qm^2 + 2q^2m + 2m^2 - qm - m + 2q - 1$.

Taking $p_1 = qm - m + 1 + mr - y$ and $p_2 = (q-1)(qm - m + 1) - (qm - m + 1)r + (q-1)y$, we know that

$$\begin{pmatrix} r \\ y \end{pmatrix} + p_1 \begin{pmatrix} q-1 \\ qm - m + 1 \end{pmatrix} + p_2 \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} (2qm - 2m + 2)(q-1) \\ (2qm - 2m + 1)(qm - m + 1) \end{pmatrix}.$$

Noting that $0 \leq r \leq q-1$ and $0 \leq y \leq qm - p + 2$. If $r = q-1$, then $y \geq 0$. If $0 \leq y \leq qm - m + 1$, then $r \geq 0$.

If $qm - m + 2 \leq y \leq qm - p + 2$, then $r \geq 1$. Obviously, $p_1 \geq 0$ and $p_2 \geq 0$. This gives

$$\begin{aligned} \exp(D) &\leq (2qm - 2m + 2)(q-1) + (2qm - 2m + 1)(qm - m + 1) \\ &= 2q^2m^2 - 4qm^2 + 2q^2m + 2m^2 - qm - m + 2q - 1. \end{aligned}$$

Case 2: $\exp(D) \geq 2q^2m^2 - 4qm^2 + 2q^2m + 2m^2 - qm - m + 2q - 1$.

At this time, there exists a continuous red path of $q-1$ length in D . Without loss of generality, we can take v_i and v_j as the starting and ending vertices of the $q-1$ consecutive red arcs, then it is decomposed into $(q-1, 0)$. So

$$MX = \begin{pmatrix} h_1 - (q-1) \\ h_2 \end{pmatrix}$$

has a solution. Therefore,

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} q-1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} -(q-1)m \\ (q-1)(qm - m + 1) \end{pmatrix}.$$

Then $u_2 \geq (q-1)(qm - m + 1)$. Next, we take v_i and v_j as the ending and starting vertices of the $q-1$ consecutive red arcs, then there are two paths from v_i to v_j , and they are decomposed into $(1, qm - p + 2)$ or $(0, qm - m + 1)$. So

$$MX = \begin{pmatrix} h_1 - 1 \\ h_2 - (qm - p + 2) \end{pmatrix}$$

or

$$MX = \begin{pmatrix} h_1 \\ h_2 - (qm - m + 1) \end{pmatrix}$$

has a solution. Therefore,

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} 1 \\ qm - p + 2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} qm - m - p + 2 \\ qm - m + 1 - (q-1)(qm - p + 2) \end{pmatrix}$$

or

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} 0 \\ qm - m + 1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} qm - m + 1 \\ -(q-1)(qm - m + 1) \end{pmatrix}.$$

Then $u_1 \geq qm - m + 1$. Thus

$$\begin{aligned} h_1 + h_2 &= (1 \ 1)M(u_1 \ u_2)^T \geq (qm - m + q \ m + 1)(qm - m + 1 \ (q-1)(qm - m + 1))^T \\ &= 2q^2m^2 - 4qm^2 + 2q^2m + 2m^2 - qm - m + 2q - 1. \end{aligned}$$

That is $\exp(D) \geq 2q^2m^2 - 4qm^2 + 2q^2m + 2m^2 - qm - m + 2q - 1$.

Case 3: If the $q-1$ red arcs are not consecutive, then

$\exp(D) < 2q^2m^2 - 4qm^2 + 2q^2m + 2m^2 - qm - m + 2q - 1$.

In this case, there are at most $q-2$ long continuous yellow paths on the $(qm - m + q)$ -cycle. Taking $p_1 = qm - m + mr - y$ and $p_2 = (qm - m + 1)(q-1) - (q-1) - (qm - m + 1)r + (q-1)y$, we see that

$$\begin{pmatrix} r \\ y \end{pmatrix} + p_1 \begin{pmatrix} q-1 \\ qm - m + 1 \end{pmatrix} + p_2 \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} (2qm - 2m)(q-1) \\ (2qm - 2m)(qm - m + 1) - m(q-1) \end{pmatrix}.$$

Noting that $0 \leq r \leq q-1$ and $0 \leq y \leq qm-p+2$. If $0 \leq r \leq q-2$, then $y \geq 0$. If $r = q-1$, then $y \geq 1$. If $0 \leq y \leq qm-m$, then $r \geq 0$. If $qm-m+1 \leq y \leq qm-p+2$, then $r \geq 1$. Obviously, $p_1 \geq 0$ and $p_2 \geq 0$. This gives

$$\begin{aligned} \exp(D) &\leq (2qm-2m)(q-1) + (2qm-2m)(qm-m+1) - m(q-1) \\ &= 2q^2m^2 - 4qm^2 + 2q^2m + 2m^2 - 3qm + m \\ &< 2q^2m^2 - 4qm^2 + 2q^2m + 2m^2 - qm - m + 2q - 1. \end{aligned}$$

Similar to the proof of theorem 3.2 and 3.3, we can also prove that all common arcs are yellow in three cases, and get the corresponding upper bound of primitive exponent.

Theorem 3.4 If $q > 1$, $p \geq 1$, the common arcs are all yellow and D is primitive, then

$$\exp(D) = 2q^2m^2 - 2qm^2 + 2q^2m - 2qpm + qm + 2pm - 2qp + 2q - m + p$$

if and only if there exists a continuous yellow path of $qm-p+1$ length in D .

D. The lower bounds of exponent and the characterization of extremal two-colored digraphs when $q > 1$

In combination with Figure.1 and Formula (3), we can see that the number of yellow arcs is m times plus 1 of the number of red arcs in the $(qm-m+q)$ -cycle. According to the Drawer Theorem, no matter how all arcs in D are colored, there is at least a yellow path of length $m+1$ in D and there is always a yellow path of length m in the $(m+1)$ -cycle. In this section, we are going to focus on the lower bounds of the primitive exponents of D .

Theorem 3.5 If $q > 1$ and D is primitive, then

$$\exp(D) = \begin{cases} 2qm^2 - 2m^2 + 4qm - 2m + 2q & (p = 0, \text{ that is, the two cycles intersect at a point}) \\ 2qm^2 - 2m^2 + 4qm - 3m + 2q - 1 & (q > 1 \text{ and the common arcs contain a red arc}) \\ 2qm^2 - 2m^2 + 4qm + qpm - pm + qp - 2m + 2q - p & (q > 1 \text{ and the common arcs do not contain a red arc}) \end{cases}$$

if and only if there is at most a yellow path of length $m+1$ in D .

Proof At this time, there exists a continuous yellow path of length $m+1$ at most in D . Without loss of generality, we can take v_i and v_j as the starting and ending vertices of the $m+1$ consecutive yellow arcs, then it is decomposed into $(0, m+1)$. So

$$MX = \begin{pmatrix} h_1 \\ h_2 - (m+1) \end{pmatrix}$$

has a solution. Therefore,

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} 0 \\ m+1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} m+1 \\ -(q-1)(m+1) \end{pmatrix}.$$

Then $u_1 \geq m+1$.

Combined with the uncolored digraph of D , we know the two cycles may intersect at a point, or the red arc in the $(m+1)$ -cycle may be on the common arc, or the common arcs are yellow in D . Therefore, the following discussion is divided into three situations.

Case 1: The two cycles intersect at a point.

Subcase 1.1: $\exp(D) \geq 2qm^2 - 2m^2 + 4qm - 2m + 2q$.

At this time, $0 \leq r \leq q$, then $0 \leq y \leq qm+1$. we can take v_i and v_j as the ending and starting vertices of the $m+1$ consecutive yellow arcs. If v_i and v_j are on the $(qm-m+q)$ -cycle, then it is decomposed into $(q-1, qm-2m)$. If v_i and v_j are not on the common arc and they are on two different cycles, then it is decomposed into $(q, qm-m)$. So

$$MX = \begin{pmatrix} h_1 - (q-1) \\ h_2 - (qm-2m) \end{pmatrix}$$

or

$$MX = \begin{pmatrix} h_1 - q \\ h_2 - (qm-m) \end{pmatrix}$$

has a solution. Therefore,

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} q-1 \\ qm-2m \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} -m \\ (q-1)(m+1) \end{pmatrix}$$

or

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} q \\ qm-m \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} -m \\ qm-m+q \end{pmatrix}.$$

Then $u_2 \geq qm-m+q$. Thus

$$\begin{aligned} h_1 + h_2 &= (1 \ 1)M(u_1 \ u_2)^T \geq (qm-m+q \ m+1) \begin{pmatrix} m+1 \\ qm-m+q \end{pmatrix} \\ &= 2qm^2 - 2m^2 + 4qm - 2m + 2q. \end{aligned}$$

That is $\exp(D) \geq 2qm^2 - 2m^2 + 4qm - 2m + 2q$.

Subcase 1.2: $\exp(D) \leq 2qm^2 - 2m^2 + 4qm - 2m + 2q$.

Taking $p_1 = m+1+mr-y$ and $p_2 = qm-m+q-(qm-m+1)r+(q-1)y$, we see that

$$\begin{pmatrix} r \\ y \end{pmatrix} + p_1 \begin{pmatrix} q-1 \\ qm-m+1 \end{pmatrix} + p_2 \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} (m+1)(q-1)+qm-m+q \\ (m+1)(qm-m+1)+m(qm-m+q) \end{pmatrix}.$$

Noting that $0 \leq r \leq q$ and $0 \leq y \leq qm+1$. Combined with the uncolored digraph of D , we know if $0 \leq y \leq m+1$, then $r \geq 0$; if $m+2 \leq y \leq 2m+1$, then $r \geq 1$; if $(x-1)m+2 \leq y \leq xm+1$ ($2 \leq x \leq q$), then $r \geq x$.

Obviously, $p_1 \geq 0$ and $p_2 \geq 0$. This gives

$$\begin{aligned} \exp(D) &\leq (m+1)(q-1)+qm-m+q+(m+1)(qm-m+1)+m(qm-m+q) \\ &= 2qm^2 - 2m^2 + 4qm - 2m + 2q. \end{aligned}$$

Subcase 1.3: If there is a shortest yellow path longer than $m+1$, then

$$\exp(D) > 2qm^2 - 2m^2 + 4qm - 2m + 2q.$$

In this case, there are at least $m+2$ long continuous yellow paths on the $(qm-m+q)$ -cycle. Taking

$p_1 = m+2+mr-y$ and $p_2 = qm-m+q-(qm-m+1)r+(q-1)y$, we see that

$$\begin{pmatrix} r \\ y \end{pmatrix} + p_1 \begin{pmatrix} q-1 \\ qm-m+1 \end{pmatrix} + p_2 \begin{pmatrix} 1 \\ m \end{pmatrix} = \begin{pmatrix} (m+2)(q-1)+qm-m+q \\ (m+2)(qm-m+1)+m(qm-m+q) \end{pmatrix}.$$

Noting that $0 \leq r \leq q$ and $0 \leq y \leq qm+1$. Combined with the uncolored digraph of D , we know if $0 \leq y \leq m+2$, then $r \geq 0$; if $m+3 \leq y \leq 2m+2$, then $r \geq 1$; if $(x-1)m+3 \leq y \leq xm+2$ ($2 \leq x \leq q$), then $r \geq x$.

Obviously, $p_1 \geq 0$ and $p_2 \geq 0$. This gives

$$\begin{aligned} \exp(D) &\leq (m+2)(q-1)+qm-m+q+(m+2)(qm-m+1)+m(qm-m+q) \\ &= 2qm^2 - 2m^2 + 5qm - 3m + 3q > 2qm^2 - 2m^2 + 4qm - 2m + 2q. \end{aligned}$$

In summary, if the two cycles intersect at a point, D is primitive and there is at most one $m+1$ long yellow path in D , then $\exp(D) = 2qm^2 - 2m^2 + 4qm - 2m + 2q$.

Case 2: The common arcs contain a red arc.

At this time, $0 \leq r \leq q-1$, then $0 \leq y \leq qm-p+2$. Based on Figure.1, we know there is an m long yellow path in the $(m+1)$ -cycle. If the starting or ending point of p_{v_i, v_j} is on $qm-m+q+1 \rightarrow qm-m+q+2 \rightarrow \dots \rightarrow$

$qm+q-p$, thus p_{v_i, v_j} must contain a red common arc, so the $m+1$ long yellow path can only be obtained in the $(qm-m+q)$ -cycle. Similar to the analysis of cases 1 in this theorem, to prove $\exp(D) = 2qm^2 - 2m^2 +$

$4qm-3m+2q-1$, we can prove it from three aspects: $\exp(D) \geq 2qm^2 - 2m^2 + 4qm - 3m + 2q - 1$, $\exp(D) \leq 2qm^2 - 2m^2 + 4qm - 3m + 2q - 1$ and the existence of continuous yellow paths longer than $m+1$, not in detail.

Case 3: The common arcs do not contain a red arc, that is, the common arcs are yellow in D .

At this time, $0 \leq r \leq q$, then $0 \leq y \leq qm - p + 1$. Based on Figure.1, we know that there is an m long yellow path in the $(m+1)$ -cycle. The starting and ending points of p_{v_i, v_j} will not necessarily be on $(m+1)$ -cycle at the same time. Similar to the analysis of cases 1 in this theorem, to prove $\exp(D) = 2qm^2 - 2m^2$

$+4qm + qpm - pm + qp - 2m + 2q - p$, we can prove it from three aspects: $\exp(D) \geq 2qm^2 - 2m^2 + 4qm + qpm -$

$pm + qp - 2m + 2q - p$, $\exp(D) \leq 2qm^2 - 2m^2 + 4qm + qpm - pm + qp - 2m + 2q - p$ and the existence of continuous yellow paths longer than $m+1$, not in detail.

IV. THE SET OF THE PRIMITIVE EXPONENT

In the previous chapter, we have classified and discussed the range of the exponents in the primitive case of the two-colored graph, as shown in Figure.1, and characterized the corresponding extreme digraphs. In this section, we will discuss the primitive exponents in other situations and characterize the set of primitive exponent.

Theorem 4.1 If $q > 1$, $p = 0$, D is primitive, then

$$\exp(D) \geq 2k_1qm - 2k_1m + 2k_1q - k_1 + m + 1 \quad (m + 2 \leq k_1 \leq qm)$$

if there exists a continuous yellow path of k_1 length in D .

Proof When $q > 1$, $p = 0$, the $(qm - m + q)$ -cycle and $(m+1)$ -cycle intersect at point $qm - m + q$. At this point, the two cycles have q red arcs and $qm + 1$ yellow arcs in D , and there exists a continuous yellow path of k_1 ($m + 2 \leq k_1 \leq qm$) length at least in D . Without loss of generality, we can take v_i and v_j as the starting and ending vertices of the k_1 consecutive yellow arcs in D , then there is only one path from v_i to v_j and it is decomposed into $(0, k_1)$. So

$$MX = \begin{pmatrix} h_1 \\ h_2 - k_1 \end{pmatrix}$$

has a solution. Therefore,

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} 0 \\ k_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} k_1 \\ -(q-1)k_1 \end{pmatrix}.$$

Then $u_1 \geq k_1$.

Next, we take v_i and v_j as the ending and starting vertices of the k_1 consecutive yellow arcs in D . If v_i and v_j are on the $(qm - m + q)$ -cycle, then there is only one path from v_i to v_j and it is decomposed into $(q-1, qm - m + 1 - k_1)$. If v_i and v_j are on two different cycles, then there is only one path from v_i to v_j and it is decomposed into $(q, qm + 1 - k_1)$. So

$$MX = \begin{pmatrix} h_1 - (q-1) \\ h_2 - (qm - m + 1 - k_1) \end{pmatrix}$$

or

$$MX = \begin{pmatrix} h_1 - q \\ h_2 - (qm + 1 - k_1) \end{pmatrix}$$

has a solution. Therefore,

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} q-1 \\ qm - m + 1 - k_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} -k_1 + 1 \\ (q-1)k_1 \end{pmatrix}$$

or

$$X = M^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} - M^{-1} \begin{pmatrix} q-1 \\ qm + 1 - k_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} -k_1 + 1 \\ qk_1 - k_1 + 1 \end{pmatrix}.$$

Then $u_2 \geq qk_1 - k_1 + 1$. Thus

$$\begin{aligned} h_1 + h_2 &= (1 \ 1)M(u_1 \ u_2)^T \geq (qm - m + q \ m + 1)(k_1 \ qk_1 - k_1 + 1)^T \\ &= 2k_1qm - 2k_1m + 2k_1q - k_1 + m + 1. \end{aligned}$$

That is $\exp(D) \geq 2k_1qm - 2k_1m + 2k_1q - k_1 + m + 1$.

Similar to theorem 4.1, we also prove theorem 4.2-4.3. Therefore, the method of proof is similar, not to give a detailed proof process.

Theorem 4.2 If $q > 1$, $p \geq 1$, the common arcs contain a red arc and D is primitive, then

$$\exp(D) \geq 2k_2qm - 2k_2m + 2k_2q - k_2 \quad (m+2 \leq k_2 \leq qm - p + 1)$$

if there exists a continuous yellow path of k_2 length in D .

Theorem 4.3 If $q > 1$, $p \geq 1$, the common arcs are yellow and D is primitive, then

$$\exp(D) \geq 2k_3qm - 2k_3m + 2k_3q + qpm - pm + m + qp - p - k_3 + 1 \quad (m+2 \leq k_3 \leq qm - p)$$

if there exists a continuous yellow path of k_3 length in D .

By combining theorem 3.1-3.5 and theorem 4.1-4.3, we can obtain the set of primitive exponents of the two-colored digraph.

Theorem 4.4 For some nonnegative integers q , $p(p \leq m)$, $m(m \geq 1)$, and D is primitive, then the set of primitive exponent is

$$\begin{aligned} & \{2m + 1\} \cup \{2qm^2 - 2m^2 + 4qm - 2m + 2q \mid q > 1\} \cup \{2qm^2 - 2m^2 + 4qm - 3m + 2q - 1 \mid q > 1\} \cup \\ & \quad \{2qm^2 - 2m^2 + 4qm + qpm - pm + qp - 2m + 2q - p \mid q > 1, p \geq 1\} \cup \\ & \quad \{2k_1qm - 2k_1m + 2k_1q - k_1 + m + 1 \mid q > 1, m + 2 \leq k_1 \leq qm\} \cup \\ & \quad \{2k_2qm - 2k_2m + 2k_2q - k_2 \mid q > 1, m + 2 \leq k_2 \leq qm - p + 1\} \cup \\ & \quad \{2k_3qm - 2k_3m + 2k_3q + qpm - pm + m + qp - p - k_3 + 1 \mid q > 1, p \geq 1, m + 2 \leq k_3 \leq qm - p\} \cup \\ & \quad \{2q^2m^2 - 2qm^2 + 2q^2m + qm - m + 2q \mid q > 1\} \cup \{2q^2m^2 - 4qm^2 + 2q^2m + 2m^2 - qm - m + 2q - 1 \mid q > 1\} \cup \\ & \quad \{2q^2m^2 - 2qm^2 + 2q^2m - 2qpm + qm + 2pm - 2qp + 2q - m + p \mid q > 1, p \geq 1\}. \end{aligned}$$

V. CONCLUSION

This paper studies the digraph D shown in Figure 1. First, D is primitive, the conditions that each element of the cycle matrix corresponding to D should meet are discussed. Secondly, we discuss the index by case and find the range of the primitive case. Thirdly, all the arcs in D are colored, and the coloring conditions of the upper and lower bounds of exponents are found. Finally, the primitive exponential set of D is obtained, that is, Theorem 4.4, which is also the conclusion of the paper. The research methods and conclusions of this paper can provide some reference for the research of the primitive exponent, connectivity index and scrambling index of edge staining of graphs in the fields of communication networks, coding cache and chemical molecular mechanism.

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