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An Algorithm for Solving Three-dimensional Assignment Problem

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Abstract: - This article presents a algorithm for solving Three-dimensional assignment problem. Firstly, decompose the three-dimensional cubic matrix corresponding to the three-dimensional assignment problem into multiple two-dimensional planar matrices, and obtain that the assignment problems corresponding to these two-dimensional planar matrices have the same feasible solution as the original three-dimensional assignment problem. Then, the leading principal submatrix algorithm is used to solve each two-dimensional assignment problem. The characteristic of the leading principal submatrix algorithm is that each operation only needs to consider the local (leading principal submatrix) of the assignment matrix of the two-dimensional assignment matrix, without considering the entire assignment matrix. Starting from the first-order leading principal submatrix of the assignment matrix, Through the same solution transformation, the row minimum element of the leading principal submatrix of each order of the assignment matrix is found step by step, and the optimal solution of the two-dimensional assignment problems are obtained. Finally, by comparing the optimal solutions of these two-dimensional assignment problems, the optimal solutions of the original three-dimensional assignment problems are obtained. This algorithm can find the optimal solution for three-dimensional assignment problems in a patterned manner, facilitating computer programming and handling assignment problems with a large number of people and tasks.

Keywords: Three-dimensional Assignment Problem, Algorithm, Optimal Solution, Same Solution Transformation, Leading Principal Submatrix, Row Minimum Element.

I. INTRODUCTION

The assignment problem is an important branch of operations research. Under existing resources, how to effectively allocate and assign tasks to achieve higher efficiency, greater returns, and lower costs is the core of assignment problem research. There are mature theories and rich algorithms for the study of two-dimensional assignment problems. In 1955, American operations researcher

H. W. Kuhn first established the Hungarian algorithm for solving n-persons and n-tasks assignment problems [1], Later scholars successively proposed the matrix reduction analysis algorithm [2] and the cut-top and exclusion algorithm [3-5] for solving balanced assignment problems, and extended the problem to the shortest time limit assignment problems [6,7], absent assignment problems [8,9], and generalized assignment problems [10-12], among others.

With the further development of modern socio-economic, medical technology, national defense technology, and other fields, more complex multidimensional (dimension greater than or equal to 3) assignment systems have emerged. Many scholars have begun to study this multidimensional assignment problem. In 1968, Pierskalla W P first proposed the multidimensional assignment problem [13] and pointed out that the three-dimensional assignment problem is an NP hard problem. Later, scholars mainly focused on the study of approximate solutions to the multidimensional assignment problem, Heuristic algorithms [14], genetic algorithms [15], and hybrid genetic algorithms [16] have been proposed for solving three-dimensional assignment problems. In recent years, research has mainly focused on the stability and asymptotic behavior of solutions [17-20].

This article proposes a algorithm for solving three-dimensional assignment problem. Firstly, decompose the three-dimensional cubic matrix corresponding to the three-dimensional assignment problem into multiple two-dimensional planar matrices, and obtain that the assignment problems corresponding to these two-dimensional pianar matrices have the same feasible solution as the original three-dimensional assignment problem, and then the leading principal submatrix algorithm is used to solve each two-dimensional assignment problem. Finally, compare the optimal solutions of these two-dimensional assignment problems to obtain the optimal solution of the original problem. The leading principal submatrix algorithm for solving the two-dimensional assignment problem is characterized by that each operation is limited to the local of the assignment matrix of the two-dimensional assignment matrix, through the same solution transformation, find the row minimum element of each order leading principal submatrix of the same solution assignment matrix step by step, and then get the optimal solution of the original problem.

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The algorithm presented in this article can be used to find the optimal solution of a three-dimensional assignment problem in a pattern. When the number of people and tasks involved in the assignment problem is large, this algorithm is convenient for computer programming and can handle more complex assignment problems. This algorithm is suitable for solving complex assignment problems in the national economy, such as communication, education, and healthcare.

II. MATHEMATICAL MODEL AND BASIC CONCEPTS

The problem description of the three-dimensional assignment problem is as follows: Let $A=\{a_1,a_2,\cdots,a_n\}$ represent a set of n people, $B=\{b_1,b_2,\cdots,b_m\}$ represent a set of m people, and $C=\{c_1,c_2,\cdots,c_p\}$ represent a set of p tasks. Now, take one person a_i from set A and one person b_j from set B to form a team to complete one task c_k in set C. The quantity indicator (cost or benefit) is recorded as d_{ijk}. It is required to complete $l\left(l\le\min\{m,n,p\}\right)$ tasks in set C, and each team can only complete one task, different teams can complete different tasks. How to form a team, assigning tasks to achieve maximum efficiency in completing tasks. This article studies the problem of minimizing the cost of balancing three-dimensional assignment problems, that is, the situation of $n=m=p=l$. The mathematical model is:

$$\min\sum_{i=1}^{l}\sum_{j=1}^{l}\sum_{k=1}^{l}d_{ijk}x_{ijk},\ s.t.\begin{cases}\sum_{i=1}^{l}\sum_{j=1}^{l}x_{ijk}=1,k=1,2,\cdots,l\\ \sum_{i=1}^{l}\sum_{k=1}^{l}x_{ijk}=1,j=1,2,\cdots,l\\ \sum_{j=1}^{l}\sum_{k=1}^{l}x_{ijk}=1,i=1,2,\cdots,l\end{cases}.$$

Where x_{ijk} is the $0-1$ variable, $x_{ijk}=1$ represents assigning the i-th person in set A and the j-th person in set B to team up to complete the k-th task in set C, otherwise $x_{ijk}=0$.

Below, we provide the concepts of assignment volume matrix, transverse plane matrix, longitudinal plane matrix, vertical plane matrix, feasible solution, optimal solution for three-dimensional assignment problems, as well as feasible solution, optimal solution for two-dimensional assignment problems.

Definition 1 Establish a spatial Cartesian coordinate system, place the cost index d_{ijk} at point (i,j,k), and the resulting cube is called the assignment cubic matrix, denoted as D. The planar matrix obtained by cutting off the assignment cubic matrix D from the plane $x=i$ is called the i-th transverse plane matrix of the assignment cubic matrix D, denoted as X_i; The planar matrix obtained by cutting off the assignment cubic matrix D in plane $y=j$ is called the j-th longitudinal plane matrix of the assignment cubic matrix D, denoted as Y_j; The planar matrix obtained by cutting off the assignment cubic matrix D in plane $z=k$ is called the k-th vertical plane matrix of the assignment cubic matrix D, denoted as Z_k.

Definition 2 Takes n elements $d_{i_p j_p k_p}$ $\left(1\le p\le n,p\in N\right)$ from different planar matrices in the assignment cubic matrix $D=\left(d_{ijk}\right)_{n\times n\times n}$, where $\left\{i_p\middle|1\le p\le n\right\}=\left\{j_p\middle|1\le p\le n\right\}=\left\{k_p\middle|1\le p\le n\right\}=\{1,2,\cdots,n\}$, we call these elements independent of each other, and call their subscript set $\left\{\left(i_p,j_p,k_p\right)\middle|1\le p\le n\right\}$ a feasible solution to the three-dimensional assignment problem determined by D.

Definition 3 Let $\left\{\left(i_p^*,j_p^*,k_p^*\right)\middle|1\le p\le n\right\}$ be a feasible solution to the three-dimensional assignment problem determined by the cubic matrix $D=\left(d_{ijk}\right)_{n\times n\times n}$. If there is $\sum_{p=1}^{n}d_{i_p j_p k_p}\ge\sum_{p=1}^{n}d_{i_p^* j_p^* k_p^*}$ for any feasible solution

$\left\{\left(i_p, j_p, k_p\right) \middle| 1 \le p \le n\right\}$ of D, then $\left\{\left(i_p^*, j_p^*, k_p^*\right) \middle| 1 \le p \le n\right\}$ is called an optimal solution to the three-dimensional assignment problem determined by D. $\sum_{p=1}^{n} d_{i_p^* j_p^* k_p^*}$ is called the optimal value of the three-dimensional assignment problem determined by D.

Definition 4 Takes n elements $a_{i_p j_p}\left(1 \le p \le n, p \in N\right)$ from different rows and columns in the assignment matrix $A = \left(a_{ij}\right)_{n\times n}$, where $\left\{i_p \middle| 1 \le p \le n\right\} = \left\{j_p \middle| 1 \le p \le n\right\} = \left\{1, 2, \cdots, n\right\}$, we call these elements independent of each other and call their subscript set $\left\{\left(i_p, j_p\right) \middle| 1 \le p \le n\right\}$ a feasible solution to the n-persons and n-tasks assignment problem determined by the matrix A.

Definition 5 Let $\left\{\left(i_p^*, j_p^*\right) \middle| 1 \le p \le n\right\}$ be a feasible solution to the n-persons and n-tasks assignment problem determined by matrix $A = \left(a_{ij}\right)_{n\times n}$. If there is $\sum_{p=1}^{n} a_{i_p j_p} \ge \sum_{p=1}^{n} a_{i_p^* j_p^*}$ for any feasible solution $\left\{\left(i_p, j_p\right) \middle| 1 \le p \le n\right\}$ of A, then $\left\{\left(i_p^*, j_p^*\right) \middle| 1 \le p \le n\right\}$ is called an optimal solution to the n-persons and n-tasks assignment problem determined by matrix A, and $\sum_{k=1}^{n} a_{i_t^* j_t^*}$ is the optimal value of the n-persons and n-tasks assignment problem determined by matrix A.

Definition 6 Let $a_{i_p j_p}\left(1 \le p \le k, p \in N\right)$ be a group of independent k elements in the assignment matrix $A = \left(a_{ij}\right)_{n\times n}$ and $\min_{j\in\{1,2,\cdots,n\}} a_{i_p j} = a_{i_p j_p}\left(1 \le p \le k\right)$, then $a_{i_p j_p}\left(1 \le p \le k, p \in N\right)$ is called a group of independent k row minimum elements in A. If there is a group of independent n row minimum elements in matrix A, then its subscript set is an optimal solution to the assignment problem of n-persons and n-tasks determined by matrix A.

III. LEMMA AND MAIN CONCLUSION

Lemma 1 Let is the assignment matrix of n-persons and n-tasks assignment problem, then adding the same constant to a row (column) of matrix yields matrix. The assignment problem determined by assignment matrix has the same optimal solution as the assignment problem determined by the original assignment matrix.

Proof From definition 5, the lemma conclusion holds.

In article [3], L.Z. Zhou gives the same solution transformation theory of the balanced assignment problem, and in article [5], he gives the principal submatrix algorithm of the n-persons and 2n-tasks assignment problem. In article [3], if the transformation is limited to the leading principal submatrix of the assignment matrix, we can obtain the leading principal submatrix algorithm of the n-persons and n-tasks assignment problem as follows:

Theorem 1 Let $A_{1,k}^{(k)}$ be the k-order leading principal submatrix of the assignment matrix $A^{(k)} = (a_{ij}^{(k)})_{n\times n}$, and $\left\{a_{i_t j_t}^{(k)}\left(1 \le t \le k, t \in N\right)\right\}$ be a group of k independent row minimum elements in $A_{1,k}^{(k)}$ (the row minimum elements are relative to the leading principal submatrix $A_{1,k}^{(k)}$), where $\left\{i_1, i_2, \cdots, i_k\right\} = \left\{j_1, j_1, \cdots, j_k\right\} = \left\{1, 2, \cdots k\right\}$, $A^{(k)}, A^{(k-1)}, \cdots A^{(0)}$ is a group of assignment matrices obtained through gradual transformation with $A^{(k)}$ as the initial matrix.

(1) The rule for transforming from $A^{(k)}$ to $A^{(k-1)}$ is: if $C_{j_k} \ge 0$, then add the constant C_{j_k} to column j_k of $A^{(k)}$ to obtain $A^{(k-1)}$; If $C_{j_k} < 0$, then subtract the constant C_{j_k} from each element in column j_{k+1} (let $j_{k+1} = k+1$, $i_{k+1} = k+1$) of $A^{(k)}$ to obtain $A^{(k-1)}$. Where C_{j_k}, i_k, and j_k are determined to satisfy:

$$C_{j_k} = \min_{1\le t\le k}\left(a_{i_t j_{k+1}}^{(k)} - a_{i_t j_t}^{(k)}\right) = a_{i_k j_{k+1}}^{(k)} - a_{i_k j_k}^{(k)} .$$

(2) The rule for transforming from $A^{(p)}$ to $A^{(p-1)}\left(p = k-1,\cdots,2,1\right)$ is to add the constant C_{j_p} to each element in column j_p of $A^{(p)}$ to obtain $A^{(p-1)}$, where C_{j_p}, i_p and j_p are determined to satisfy:

$$C_{j_p} = \min_{1\le t\le p}\left(\min_{j\in\left\{j_{p+1},j_{p+2}\cdots,j_{k+1}\right\}} a_{i_t j}^{(p)} - a_{i_t j_t}^{(p)}\right) = a_{i_p j_s}^{(p)} - a_{i_p j_p}^{(p)}\ \left(j_s \in \left\{j_{p+1}, j_{p+2},\cdots, j_{k+1}\right\}\right).$$

Then there must be $k+1$ independent row minimum elements can be found in the $k+1$-order leading principal submatrix $A_{1,k+1}^{(0)}$ of $A^{(0)}$,which is gradually derived from the initial matrix $A^{(k)}$ according to this rule.

Proof The proof can be referred to article [3, 5]. For the convenience of reading, another proof is given below, which is carried out in two steps.

Step 1. Using Mathematical induction to prove the $k+1$-order leading principal submatrix $A_{1,k+1}^{(0)}$ of the assignment matrix $A^{(0)}$, has the following characteristics:

(a) $\left\{a_{i_t j_t}^{(0)}\left(1\le t\le k, t\in N\right)\right\}$ be a group of k independent row minimum elements in the leading principal submatrix $A_{1,k+1}^{(0)}$.

(b) $\min_{j\in\left\{j_{t+1},j_{t+2}\cdots,j_{k+1}\right\}} a_{i_t j}^{(0)} = a_{i_t, j_t}^{(0)}\ \left(t = 1,2,\cdots,k\right)$.

From the definition of independent row minimum element, it is obvious that the first step in the theorem is the transformation from $A^{(k)}$ to $A^{(k-1)}$, we can determine that $\left\{a_{i_t j_t}^{(k-1)}\left(1\le t\le k, t\in N\right)\right\}$ be a group of k independent row minimum elements in $A_{1,k+1}^{(k-1)}$, and there is $a_{i_k j_{k+1}}^{(k-1)} = a_{i_k, j_k}^{(k-1)}$.

Suppose the transformation conclusion from $A^{(s)}$ to $A^{(s-1)}\left(1\le s\le k-1\right)$ holds, that is, $\left\{a_{i_t j_t}^{(s-1)}\left(1\le t\le k, t\in N\right)\right\}$ be a group of k independent row minimum elements in $A_{1,k+1}^{(s-1)}$, and there is $\min_{j\in\{j_{t+1},j_{t+2},\cdots,j_{k+1}\}} a_{i_t j}^{(s-1)} = a_{i_t, j_t}^{(s-1)}\ \left(t = s, s+1,\cdots,k\right)$.

Next, consider the transformation from $A^{(s-1)}$ to $A^{(s-2)}$, that is, add the constant $C_{j_{s-1}}$ to each element of column j_{s-1} of matrix $A^{(s-1)}$ to get $A^{(s-2)}$, then $a_{i_{s-1} j_{s-1}}^{(s-2)}$ is the row minimum element of the row i_{s-1} in $A_{1,k+1}^{(s-2)}$, and there is $\min_{j\in\{j_s,j_{s+1},\cdots,j_{k+1}\}} a_{i_{s-1} j}^{(s-2)} = a_{i_{s-1}, j_{s-1}}^{(s-2)}$. Combined with the previous assumptions, there is $\left\{a_{i_t j_t}^{(s-2)}\left(1\le t\le k, t\in N\right)\right\}$ be a group of k independent row minimum elements in $A_{1,k+1}^{(s-1)}$, and there is $\min_{j\in\{j_{t+1},j_{t+2},\cdots,j_{k+1}\}} a_{i_t j}^{(k-1)} = a_{i_t, j_t}^{(k-1)}\ \left(t = s-1, s,\cdots,k\right)$.

That is, the conclusion is also true for the transformation from $A^{(s-1)}$ to $A^{(s-2)}$. In conclusion, the conclusion (a), (b) is true!

Step2. Using Mathematical induction to prove the theorem conclusion.

The conclusion clearly holds when $k = 1$.

Suppose that the conclusion is true when $k = s$, that is, $\left\{a_{i_t j_t}^{(0)}\left(1\le t\le s, t\in N\right)\right\}$ be a group of s independent row minimum elements in $A_{1,s+1}^{(0)}$, where $\left\{i_1,\cdots,i_s\right\} = \left\{1,2,\cdots s\right\}$, $\left\{j_1,\cdots,j_{s+1}\right\} = \left\{1,2,\cdots s+1\right\}$, there is

$\min_{j\in\{j_{t+1},j_{t+2\cdots},j_{k+1}\}} a_{i_t j}^{(0)} = a_{i_t j_t}^{(0)}\left(t=1,2,\cdots,s\right)$.Then the $s+1$ independent row minimum elements can be found in the $s+1$-order leading principal submatrix $A_{1,s+1}^{(0)}$ of $A^{(0)}$.

Let's consider the $s+2$ -order leading principal submatrix $A_{1,s+2}^{(0)}$ of matrix $A^{(0)}$, if $a_{i_{s+2}j_1}^{(0)}$ is the row minimum element of row $i_{s+2}\left(i_{s+2}=s+2\right)$ of $A_{1,s+2}^{(0)}$, consider that the matrix of row i_{s+2} and column j_1 is deleted from the matrix $A^{(0)}$, and recorded as B. For convenience, the elements in matrix B still use the subscript in matrix $A^{(0)}$, then $\left\{a_{i_t j_t}^{(0)}\left(1\le t\le s+1,t\in N\right)\right\}$ be a group of s independent row minimum elements in the $s+1$-order leading principal submatrix $B_{1,s+1}$ of B, and $\min_{j\in\{j_{q+1},j_{q+2\cdots},j_{s+2}\}} a_{i_q j}^{(0)} = a_{i_q j_q}^{(0)}\left(q=2,3,\cdots,s+1\right)$, from the assumption, we can find $s+1$ independent row minimum elements in the leading principal submatrix $B_{1,s+1}$, these row minimum elements plus $a_{i_{s+2}j_1}^{(0)}$, then it is the $s+2$ independent row minimum elements of the leading principal submatrix $A_{1,s+2}^{(0)}$.

If $a_{i_{s+2}j_1}^{(0)}$ is not the minimum element in row i_{s+2} of $A_{1,s+2}^{(0)}$, consider that the matrix in row i_1 and column j_1 is removed from matrix $A^{(0)}$, and the conclusion can be reached in the same way, that is, when $k=s+1$ is reached, the conclusion is true. To sum up, the conclusion of the theorem is true!

Theorem 1 give the same solution transformation theory for solving the two-dimensional balanced assignment problem by using the leading principal submatrix algorithm, and also give the solution idea for solving the assignment problem by using the leading principal submatrix algorithm. Generally, we can start from the first order leading principal submatrix $A_{1,1}$ of the assignment matrix $A=(a_{ij})_{n\times n}$, obviously, a_{11} is the row minimum element of the first order leading principal submatrix $A_{1\times 1}$ of the assignment matrix A, According to the same solution transformation in Theorem 1, obtain the independent row minimum elements of the leading principal submatrix of each order of the same solution matrix in sequence, through the same solution transformation of degree $n-1$, finally, the n independent row minimum elements in the assignment matrix $A^{(n-1)}=(a_{ij}^{(n-1)})_{n\times n}$ with the same solution are found in a patterned way, so as to obtain the optimal solution of the original two-dimensional balanced assignment problem.

Theorem 2 let $D=\left(d_{ijk}\right)_{n\times n\times n}$ be the assignment cubic matrix to the three-dimensional assignment problem, and use planes $x=p\left(1\le p\le n\right)$ and $y=q\left(1\le q\le n\right)$ to truncate the assignment cubic matrix D, resulting in n elements with data columns denoted as H_{pq}. Take n such data columns $H_{p_s q_s}$ as columns to construct an n-order matrix A. If $\left\{p_s\middle|1\le s\le n\right\}=\left\{q_s\middle|1\le s\le n\right\}=\{1,2,\cdots,n\}$, then the three-dimensional assignment problem determined by the assignment cubic matrix D has the same feasible solution as the n-persons and n-tasks assignment problem determined by the $n!$ assignment matrices A constructed in this way.

Proof Due to the fact that $\left\{p_s\middle|1\le s\le n\right\}=\left\{q_s\middle|1\le s\le n\right\}=\{1,2,\cdots,n\}$, that is, these data columns $H_{p_s q_s}$ come from different transverse plane matrices and different longitudinal plane matrices, and based on the definition 5 mentioned earlier, it can be seen that any feasible solution to the n-persons and n-tasks assignment problem determined by these matrices A comes from different columns, that is, these feasible solutions are also located in different vertical plane matrices in D, according to the definition 2, any feasible solution of the n-persons and n-tasks assignment problem determined by these matrices A can be obtained as the feasible solution of the three-dimensional assignment problem determined by the assignment cubic matrix D.

On the contrary, based on the definition 2, it can be concluded that any feasible solution for the three-dimensional assignment problem determined by the assignment cubic matrix D comes from different transverse plane matrices, different longitudinal plane matrices, and different vertical plane matrices in D. By the construction algorithm of assignment matrix A, it can be inferred that any feasible solution for the three-dimensional assignment problem determined by the assignment cubic matrix D must be located in different rows and columns of such n-order square matrix A, The feasible solution to the n-persons and n-tasks assignment problem determined by A is verified!

Theorem 2 provides a algorithm for solving three-dimensional assignment problems. Firstly, $n!$ two-dimensional assignment matrices A are constructed from the three-dimensional assignment cubic matrix D. Then, the leading principal submatrix algorithm is used to solve each two-dimensional assignment matrix A. Finally, by comparing these optimal solutions, the solution with the minimu m optimal value is found, which is the optimal solution of the original three-dimensional assignment problem.

IV. ALGORITHM EXAMPLE

The following example illustrate the specific operations. Let's assume that the three-dimensional cubic matrix of the three-dimensional assignment problem is laid horizontally in the transverse plane matrix as follows (with the longitudinal plane matrix vertically and the vertical plane matrix horizontally): let

$$D=\begin{pmatrix}3&5&7\\4&6&2\\4&5&5\end{pmatrix}\begin{pmatrix}7&2&4\\5&2&3\\9&7&4\end{pmatrix}\begin{pmatrix}10&6&8\\7&5&2\\1&4&6\end{pmatrix},$$

Step1 Take the first column from the transverse plane matrix X_1, the second column from the transverse plane matrix X_2, and the third column from the transverse plane matrix X_3 to form the assignment matrix $A_{123}=\begin{pmatrix}3&2&8\\4&2&2\\4&7&6\end{pmatrix}$. Use the leading principal submatrix algorithm to solve the problem as follow:

$$A_{123}=\begin{pmatrix}\boxed{3}&2&8\\4&2&2\\4&7&6\end{pmatrix}2-3=-1\xrightarrow{c_2-(-1)}\begin{pmatrix}3&3&8\\4&3&2\\4&8&6\end{pmatrix}=\begin{pmatrix}\boxed{3}&3&8\\4&\boxed{3}&2\\4&8&6\end{pmatrix}\begin{matrix}8-3=5\\2-3=-1\\ \end{matrix}$$

$$\xrightarrow{c_3-(-1)}\begin{pmatrix}\boxed{3}&3&9\\4&3&3\\4&8&7\end{pmatrix}3-3=0\xrightarrow{c_1+0}\begin{pmatrix}3&3&9\\4&3&3\\4&8&7\end{pmatrix}=\begin{pmatrix}3&\boxed{3}&9\\4&3&\boxed{3}\\\boxed{4}&8&7\end{pmatrix}.$$

The optimal solution to the assignment problem corresponding to assignment matrix A_{123} is $\{(1,2),(2,3),(3,1)\}$, and the optimal value is $2+2+4=8$.

Step 2 Take the first column from the transverse plane matrix X_1, the third column from the transverse plane matrix X_2, and the second column from the transverse plane matrix X_3 to form the assignment matrix A_{132}. By using the leading principal submatrix algorithm to solve A_{132}, The optimal solution to the assignment problem corresponding to assignment matrix A_{132} is $\{(1,1),(2,3),(3,2)\}$, and the optimal value is $3+2+3=8$.

Step 3 Take the second column from the transverse plane matrix X_1, the first column from the transverse plane matrix X_2, and the third column from the transverse plane matrix X_3 to form the assignment matrix A_{213}. By using the leading principal submatrix algorithm to solve A_{213}, The optimal solution to the assignment problem corresponding to assignment matrix A_{213} is $\{(1,2),(2,3),(3,1)\}$, and the optimal value is $4+2+4=10$.

Step 4 Take the second column from the transverse plane matrix X_1, the third column from the transverse plane matrix X_2, and the first column from the transverse plane matrix X_3 to obtain the assignment matrix A_{231}. By using the leading principal submatrix algorithm to solve A_{231}, The optimal solution to the assignment problem corresponding to assignment matrix A_{231} is $\{(1,1),(2,2),(3,3)\}$, and the optimal value is $5+3+1=9$.

Step 5 Take the third column from the transverse plane matrix X_1, the first column from the transverse plane matrix X_2, and the second column from the transverse plane matrix X_3 to form the assignment matrix A_{312}. By using the leading principal submatrix algorithm to solve A_{312}, The optimal solution to the assignment problem corresponding to assignment matrix A_{312} is $\{(1,2),(2,1),(3,3)\}$, and the optimal value is $7+2+4=13$.

Step 6 Take the third column from the transverse plane matrix X_1, the second column from the transverse plane matrix X_2, and the first column from the transverse plane matrix X_3 to form the assignment matrix A_{321}. By using the leading principal submatrix algorithm to solve A_{321}, The optimal solution to the assignment problem corresponding to assignment matrix A_{321} is $\{(1,2),(2,1),(3,3)\}$, and the optimal value is $2+2+1=5$.

By comparing the optimal values of these n-persons and n-tasks assignment problems, We found that the assignment problem corresponding to assignment matrix A_{321} has the smallest optimal value, which is the optimal value of the three-dimensional assignment problem being solved, that is, $\{(1,3,2),(2,2,1),(3,1,3)\}$ is an optimal solution to the original three-dimensional assignment problem, and its optimal value is $d_{132}+d_{221}+d_{313}=2+2+1=5.$

V. CONCLUSION

This article establishes a algorithm for solving three-dimensional assignment problems. The algorithm first decomposes the original problem into multiple n-persons and n-tasks assignment problems, and then uses the leading principal submatrix algorithm to solve each n-persons and n-tasks assignment problem. Finally, the solutions of these n-persons and n-tasks assignment problems are compared to obtain the solutions of the original problem. This processing algorithm reduces complex high-dimensional assignment problems to low-dimensional assignment problems, providing a way to solve the problem. The characteristic of this leading principal submatrix algorithm for solving n-persons and n-tasks assignment problems is that each operation only needs to consider the local (leading principal submatrix) of the assignment matrix of the n-persons and n-tasks assignment problem, and does not need to consider the whole of the assignment matrix. Starting from the first order leading principal submatrix of the assignment matrix, through the same solution transformation, the row minimum element of the leading principal submatrix of each order of the assignment matrix is found step by step, and finally obtain the optimal solution of the n-persons and n-tasks assignment problem in a patterned manner. The solution algorithm proposed in this article has strong regularity, is convenient for computer programming, and can handle assignment problems with a large number of people and tasks. The next step is to further optimize the algorithm, improve computational efficiency, and solve complex assignment problems more quickly.

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