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# Efficient 2-Distance Coloring Method for Maximum and Average Sparse Graphs in Resource Allocation Optimization of Microgrids

***Abstract: -*** In microgrids management, optimizing resource allocation and enhancing efficiency are critical challenges. This paper introduces the 2-distance coloring technique as a novel approach to address these issues by focusing on the coloring problem of sparse graphs. Through rigorous theoretical derivation, we establish the minimum number of colors required for 2-distance coloring in sparse graph scenarios. Specifically, we demonstrate that for a graph G with an average maximum degree less than 2+17/20 and a maximum degree of 6, it can be list 2-distance colored using no more than 11 colors. This means that, given any set of 11 colors for each vertex, a valid 2-distance coloring can be achieved, ensuring no two adjacent vertices share the same color. This finding is pivotal, as it sets a precedent for the number of resources needed to efficiently manage and allocate resources in microgrids systems through 2-distance coloring. The application of this technique in microgrids promises to revolutionize resource distribution, reducing overlap and maximizing efficiency across the network.

***Keywords:*** 2-Distance Coloring, Colorability, Microgrids, List Coloring, Sparse Graphs.

## I. INTRODUCTION

In this study, we focus on the optimization of resource allocation within microgrids, a crucial aspect for enhancing the efficiency and reliability of distributed energy systems. Given the complexity of such networks, we limit our analysis to undirected graphs, which serve as a simplified model for microgrid connections. For the purposes of our discussion, we designate $V(G)$ as the point set, $E(G)$ as the edge set, denote the maximum degree as $\Delta(G)$, denote $d_G(x)$ as the degree of x in graph $G$ [1-5]. In graph $G$, the set of neighbors of point $v$ is referred to as $N_G(v)$. For convenience, if the graph $G$ is the same in the following discussion and does not cause misunderstandings, there is no need to write the subscript $G$. If the maximum degree of a point is $k$, it is called a $k^+$-vertex point. Similarly, if the minimum degree of a point is $k$-vertex, it is called a $k^-$-vertex point. For a sparse graph $G$, the maximum mean degree is denoted as $mad(G)$ and satisfies the condition that $mad(G) = max\{\frac{2|E(H)|}{|V(H)|}$. In this article, there are other less common symbols that are consistent with the notation in Bondy and Murty's book [6-10].

Distance coloring refers to the process of coloring any two points in graph $G$ with a distance of less than or equal to 2 with different colors in the color set. We will record the minimum number of colors as $\chi_2(G)$. List coloring refers to the fact that each point in graph $G$ has its own coloring list, and we can only choose the colors in its list to color this point. If the list of each point in a graph can be colored with appropriate colors in a way that the number of colors in the list is less than or equal to $k$, we color this list as a $k$-list coloring. In the following study, we combined 2-distance coloring with $k$-list coloring to obtain a new way of 2-distance $k$-list coloring [11-14].

In 1977, Wenger proved the following conjecture.

***Conjecture*** 1 [15] In planar graph, there is $\chi_2(G) \le 7$ when $\Delta(G) = 3$, $\chi_2(G) \le \Delta(G) + 5$ when $4 \le \Delta(G) \le 7$, and $\chi_2(G) \le 1 + \frac{3}{2}\Delta(G)$ otherwise.

However, the Conjecture 1 above is not fully proven. Many scholars have carried out further research on it [16-20].

This paper will obtain the conclusion of Theorem 1. We will make a rigorous theoretical proof by utilizing the relevant theories of graph theory.

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***Theorem 1*** If $G$ with $\mathrm{V}=6$ and $mad(G) < 2+\frac{17}{20}$, there is $\chi_2^l(G) \le 11$.

This paper primarily investigates the 2-distance coloring problem in the context of sparse graphs. The findings of this research establish a limit for the number of colors needed for the 2-distance coloring of sparse graph classes, which holds considerable importance in the realm of microgrid resource allocation. Utilizing graph theory and network analysis techniques, this approach can be adapted to address various challenges, including the optimization of resource distribution within microgrids. In microgrids, nodes can represent distribution points or energy resources, and edges can signify connections or potential interactions between these points. By applying 2-distance coloring, one can enhance the efficiency of resource allocation and utilization across the network, ensuring that similar or closely located resources are not overburdened or cause interference with each other. Moreover, this technique can aid in the scheduling of energy production and distribution, reducing conflicts and maximizing overall grid efficiency. The ability to model and analyze microgrid networks as sparse graphs and apply 2-distance coloring offers a novel approach to optimizing the operation and reliability of these energy systems. In summary, the effective application of 2-distance coloring in sparse graphs with maximum and average degrees presents a vast potential in the field of microgrids, promising enhancements in energy distribution, grid efficiency, and the sustainable management of resources.

The structure of this paper is as follows. In Section I, some basic definitions and the current research status of 2-distance coloring were introduced, and the main research objects of this paper were given; In Section II, explain the definitions that need to be used in the proof process; In Section III, the connection structures of the 2, 3, and 4 degree points were analyzed one by one, and Theorem 1 was proven using the method of proof to the contrary and transfer weights; In Section IV, it is the conclusion and prospect of this paper.

## II. NOTATION

Let's first standardize the notation in the process of proving the correctness of Theorem 1. We mark the neighbors of a $k$-vertices point as a vertex $v$, which is the set of all points adjacent to vertex $v$. Since vertex v is a $k$-vertices point, it has k adjacent points marked as $v_1, v_2, \cdots, v_{d(v)}$, for the convenience of proof, let's assume $d(v_1) \le d(v_2) \le \cdots \le d(v_{d(v)})$.

## III. PROOF OF THEOREM 1

In this section, we will rigorously prove Theorem 1. We assume that graph $G$ satisfies $\Delta = 6$, $mad(G) < 2+\frac{17}{20}$, $\chi_2^l(G) > 11$, and its $|V(G)|+|E(G)|$ sum has the smallest value, that is, $G$ is a minimal graph that satisfies the above conditions. We will use the discharge method for weight transfer to find the contradiction proof Theorem 1. The contradiction here refers to the fact that we define a primitive weight value for each point in graph $G$, and the primitive weight $\mu^*(v)$ is actually the degree of each point in the graph itself. In the following proof, we define a new method of weight transfer, so that each point has a final weight value. Finally, after transferring the new weights $\mu^*(v) \ge 2+\frac{17}{20}$ of the points in the graph, there is a contradiction between $mad(G) < 2+\frac{17}{20}$, thus obtaining Theorem 1.

Before proving Theorem 1, let's first prove the following lemmas about the minimal graph $G$.

### *A. Relevant Lemma Proof*

***Claim 1*** $\delta \ge 2$.

***Proof*** We first assume that there is a 1-degree point $v$ in the graph, and $N_G(v)$ is a neighbor of $u$, so that $\bar{G} = G - \{v\}$. We know that $G$ is a minimal graph, so $G$ must be stained by an 11-2 distance list, and the maximum degree $\Delta$ in the graph is 6. Therefore, we can definitely find a color in the list that is different from the neighbor of point $v$ and dye this 1-degree point reasonably. This contradicts our hypothesis and leads to the conclusion that $\delta < 2$, shown as Fig.1.

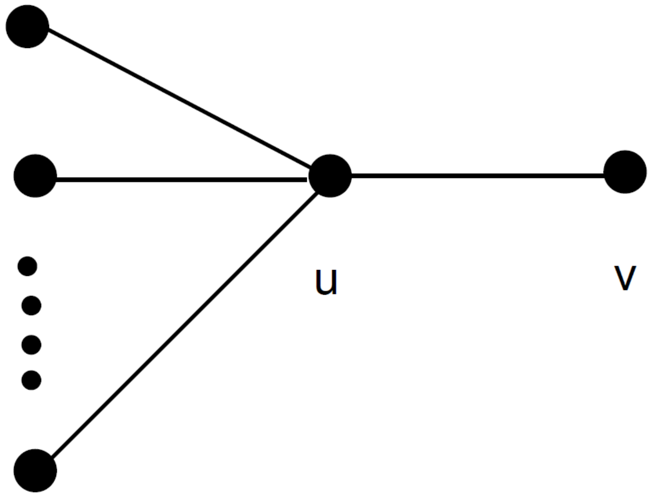


Figure 1-A list 11-2-distance Coloring of $G$

***Claim 2*** A 2-vertex can not adjacent to 2-vertices.

***Proof*** Similar to the previous proof process, because of the inequality $|F(u)|=d(x)-1+|c(x)|+|c(y)|=d(x)+1\leq\Delta+1=7$ , $|F(v)|=d(y)-1+|c(y)|+|c(u)|+|c(x)|=d(y)+2\leq\Delta+2=8$ ,we can color $u$ . Then $c$ is extended to a list 11-2-distance coloring of $G$ , a contradiction, shown as Fig.2.

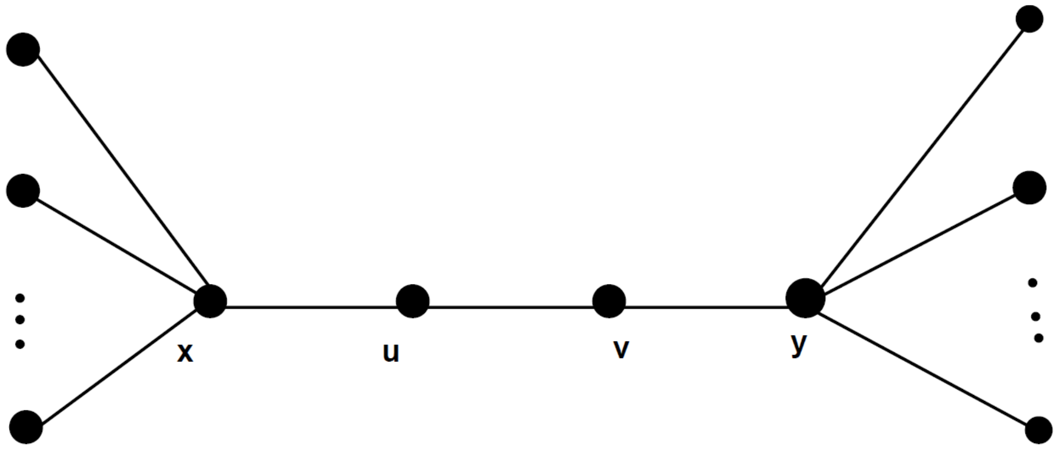


Figure 2-An Extended List 11-2-distance Coloring of $G$

***Claim 3*** Let 3-vertex $v=(v_1,v_2,v_3)$ .

***Claim 3.1*** 3-vertex cannot adjacent to three 2-vertices.

***Claim 3.2*** 3-vertex cannot adjacent to two 2-vertices.

***Claim 3.3*** If $d(v_1)=2$ , then $d(v_2)+d(v_3)\geq 10$ .

***Proof*** The proof process of Claim 3.1 can refer to Claim 2.

Then we prove Claim 3.2. Assume the contrary that $d(v_1)=d(v_2)=2$ and $d(v_3)\neq 2$ . Let $N_G(v_1)\quad\{v\}=\{x\}$ , $N_G(v_2)\quad\{v\}=\{y\}$ . Without loss of generality, let $\bar{G}=G-vv_1$ . Then $c$ is extended to a list 11-2-distance coloring of $G$ , a contradiction, shown as Fig.3.

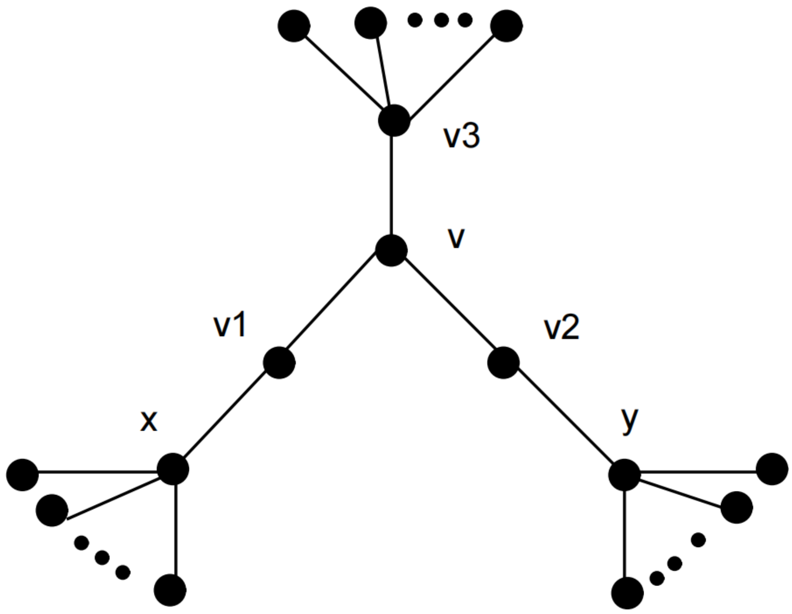


Figure 3-The Extended List 11-2-distance Coloring of $G$

The proof process of Claim 3.3 can refer to Claim 3.2. We won't go into detail here.

***Claim 4*** Let 4-vertex $v=(v_1,v_2,v_3,v_4)$,

***Claim 4.1*** 4-vertex cannot adjacent with four 2-vertices.

***Claim 4.2*** If $d(v_1)=d(v_2)=d(v_3)=2$, then $d(v_4)=6$.

Proof The proof process of Claim 4 can refer to Claim 3.

*B. Rules for Transfer of Weight*

Below, we will use the following weight transfer method to prove Theorem 1. We have designed the following weight transfer method and ensured that the weight values of all points in the graph have not changed during the weight transfer process. After all transfers, we found that the weights of all points were greater than or equal to 2+17/20, which contradicts $mad(G)<2+\frac{17}{20}$, indicating that the proof is correct.

The rules for transferring rights are:

- Rule 1: Transfer weights $\frac{21}{40}$ from each 6-degree point to adjacent points
- Rule 1: Transfer weights $\frac{43}{100}$ from each 5-degree point to adjacent points
- Rule 1: Transfer weights $\frac{67}{120}$ from each 4-degree point to adjacent points
- Rule 1: Transfer weights $\frac{17}{40}$ from each 3-degree point to adjacent points

Next, it is proven that the following relationship exists, $\mu^*(v)\geq 2+\frac{17}{20}$ , where $v\in V(G)$.

- (1) $d(v)=6$. By R1, $\mu^*(v)\geq 6-\frac{21}{40}\times 6=6-\frac{63}{20}=2+\frac{17}{20}$.
- (2) $d(v)=5$. By R2, $\mu^*(v)\geq 5-\frac{43}{100}\times 5=5-\frac{43}{20}=2+\frac{17}{20}$.
- (3) $d(v)=4$. According to Claim 4, $n_2(v)\leq 3$. We may consider two cases:

Case 1 $n_2(v)\leq 2$. In the worst case, $\mu^*(v)\geq 4-\frac{67}{120}\times 2=4-\frac{67}{60}=2+\frac{23}{30}>2+\frac{17}{20}$ , by R3.

Case 2 $n_2(v)=3$. By R1, R3, $\mu^*(v)\geq 4-\frac{67}{120}\times 3+\frac{21}{40}=4-\frac{67}{40}+\frac{21}{40}=2+\frac{17}{20}$.

- (4) $d(v)=3$. According to Claim 3, We may consider four cases:

Case 1 $v=(2,4,6)$. By R1, R3, $\mu^*(v)\geq 3-\frac{17}{40}+\frac{21}{40}>2+\frac{17}{20}$.

Case 2 $v=(2,5,5)$. By R2, $\mu^*(v)\geq 3-\frac{17}{40}+\frac{43}{100}\times 2>2+\frac{17}{20}$.

Case 3 $v=(2,5,6)$. By R1, R2, $\mu^*(v)\geq 3-\frac{17}{40}+\frac{43}{100}+\frac{21}{40}>2+\frac{17}{20}$.

Case 4 $v=(2,6,6)$. By R1, $\mu^*(v)\geq 3-\frac{17}{40}+\frac{21}{40}\times 2>2+\frac{17}{20}$.

- (5) $d(v)=2$. According to Claim 2, we may discuss ten cases, respectively.

Case 1 $v=(3,3)$. By R4, $\mu^*(v)\geq 2+\frac{17}{40}\times 2>2+\frac{17}{20}$.

Case 2 $v=(3,4)$. By R3, R4, $\mu^*(v)\geq 2+\frac{17}{40}+\frac{67}{120}>2+\frac{17}{20}$.

Case 3 $v=(3,5)$. By R2, R4, $\mu^*(v)\geq 2+\frac{17}{40}+\frac{43}{100}>2+\frac{17}{20}$.

Case 4 $v=(3,6)$. By R1, R4, $\mu^*(v)\geq 2+\frac{17}{20}+\frac{21}{40}>2+\frac{17}{20}$.

Case 5 $v=(4,4)$. By R3, $\mu^*(v)\geq 2+\frac{67}{120}\times 2>2+\frac{17}{20}$.

Case 6 $v=(4,5)$. By R2, R3, $\mu^*(v)\geq 2+\frac{67}{120}+\frac{43}{100}>2+\frac{17}{20}$.

Case 7 $v=(4,6)$. By R1, R3, $\mu^*(v)\geq 2+\frac{67}{120}+\frac{21}{40}>2+\frac{17}{20}$.

Case 8 $v=(5,5)$. By R2, $\mu^*(v)\geq 2+\frac{43}{100}\times 2>2+\frac{17}{20}$.

Case 9 $v=(5,6)$. By R1, R2, $\mu^*(v)\geq 2+\frac{43}{100}+\frac{21}{40}>2+\frac{17}{20}$.

Case 10 $v=(6,6)$. By R1, $\mu^*(v)\geq 2+\frac{21}{40}\times 2>2+\frac{17}{20}$.

The proof is completed.

## IV. CONCLUSIONS

In this paper, the authors investigate the list 2-distance chromatic number of graphs, aiming to find the minimum number of colors required to 2-distance color a given graph. The authors prove that graphs with maximum average degree less than $2+\frac{17}{20}$ and maximum degree of 6 are list 2-distance 11-colorable. This result is significant as it provides a bound on the number of colors needed to 2-distance color certain classes of sparse graphs. Moreover, the result obtained in this paper opens up new possibilities for further research in graph coloring. The techniques used in the proof could be extended to investigate other variants of the coloring problem in graphs, providing more insights into the complexity and limits of graph coloring algorithms. Furthermore, the bound on the list 2-distance chromatic number established in this paper may serve as a basis for developing efficient algorithms for coloring sparse graphs with similar properties. These algorithms could have significant practical applications in areas such as computer networks, scheduling, and resource allocation, where graph coloring is a fundamental problem.

The effective application of the 2-distance coloring technique in sparse graphs has captured interest for its potential in optimizing resource allocation within microgrids. Microgrid challenges, such as energy distribution, load balancing, and infrastructure resilience, can be modeled using graph theory. Moving forward, the adaptation of the 2-distance coloring technique in microgrid management is poised for further developments. Custom algorithms will be crafted to tackle specific microgrid optimization challenges. A key focus will be the efficient processing of large-scale data, leveraging parallel computing and distributed algorithms to manage the complex graph data inherent to microgrid networks. Research will also delve into accommodating real-time updates in dynamic microgrid environments and employing machine learning to improve forecasting and operational efficiency. This evolution aims to drive innovation in microgrid management and expand its applications into wider energy systems. These areas will constitute the main direction of the author's future research endeavors.

## ACKNOWLEDGMENT

This work was supported by the Shandong Provincial Natural Science Foundation (grant numbers ZR2023QF092).

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