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Improved Approach of Stability Domain Determination for Nonlinear Discrete Polynomial System

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Abstract- *This paper proposes an improvement of an existing algebraic method for estimating the domain of asymptotic stability of nonlinear discrete systems. The studied approach is essentially based on a polynomial description of nonlinear dynamical process and on analytic and iterative procedure of stability radius determination of nonlinear discrete systems.*

Keywords: nonlinear systems, discrete system, stability domain, polynomial description.

1. INTRODUCTION

Although the determination of the stability region is always an interesting topic in the fields of the theory of control for physical systems, works related to this topic are generally limited to a simple extension of the domain and not seeking for an exact area. For this reason, in recent years, a number of research works have been conducted in order to improve the efficiencies of the existing methods ([9], [17], [18], [20]).

Several researches have been directed for years to estimate stability domains of nonlinear continuous systems. This methods fall into two different categories. Those which use the candidate Lyapunov functions for the expansion of an invariant area around the stable equilibrium point known as Lyapunov methods ([6], [7], [8]). In fact, improvement of these techniques focus on searching for new parametric Lyapunov functions to obtain larger domain of stability of nonlinear continuous systems by using optimization techniques. The second category is based on geometrical and topological concepts for estimating a large stability domain. It is to study the behaviour of trajectories around the stable equilibrium. These methods are known as Non Lyapunov methods ([5], [13], [19]).

The richness of studies is concerned with the estimation of stability domains of nonlinear continuous systems rather than that discrete nonlinear system [10]. In addition, few studies have been made for the determination or enlargement of stability domain of discrete nonlinear systems. The most important are those in ([1], [2], [3]).

This paper is interested by studying the method related to the estimation of the stability region of discrete nonlinear systems. These systems are illustrated in the fields of electronics, computers and robotics. The methods which will be presented in this paper is Non Lyapunov technique based on the kronecker product and the Gronwell-Bellman lemma for estimating a largest domain of attraction for a particular case of nonlinear polynomial discrete systems ([14], [15]). As a new contribution, we present a functional algorithm of the developed method which is applied in physical systems in order to improve its efficiency.

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This paper is organized as follows: In the first part a description of the studied system is presented, and then a method for estimating a guaranteed stability domain is developed. In the third part a technique of enlargement of the guaranteed stability domain is exposed and finally an application of the developed algorithm to an illustrative physical third order system is presented.

2. Description of the studied polynomial discrete-time system

In this paper, we consider the discrete nonlinear systems described by the following polynomial state:

$$X_{k+1} = F(X_k) = \sum_{i=1}^{q} F_i X_k^i \tag{1}$$

Where $F_{i,} i = 1, \cdots, r$ are $(n \times n^i)$ matrices and X_k is an n-dimensional discrete-time state vector, q is the truncation order and $X_k^{[i]}$ designates the ith order kronecker power of the state $X_k^{[i]}$. The initial state is denoted by X_0. The linear part of the system (1) is asymptotically stable, it means that F_1 is a Schur matrix, i.e. the norm of eigenvalues are assumed less than one ($\|eig(F_1)\| < 1$).

3. Guaranteed stability region (GSR)

The aim is to estimate a sufficient region Ω_0 containing all initial conditions, where the equilibrium asymptotic stability $X = 0$ of the system (1) is guaranteed. The following definition explains the above description:

$$\begin{gathered} \forall X_0 \in \Omega_0, \forall k \in N, X(k, k_0, X_0) \in \Omega_0 \\ and \lim(k, k_0, X_0) = 0 \end{gathered} \tag{2}$$

$X(k, k_0, X_0)$ is the solution of equation (1) and the initial condition is defined by $X(k_0) = X_0$.

The domain of stability that we consider is defined as a ball of radius R_0 and of center the origin. The proposed set of stability is described by:

$$\Omega_0 = \{X_0 \in R^n; \|X_0\| < R_0\} \tag{3}$$

R_0 is named a stability radius of the considered system.

Lemma 1: Consider the below model description of the studied discrete nonlinear system:

$$X(k+1) = F_1.X(k) + g(k, X(k)) \tag{4}$$

We assume that the linear part is stable, the remaining part $g(k, X(k))$, which is nonlinear, verifies the following inequality:

$$g(k, X(k)) \le \beta \|X(k)\| \tag{5}$$

β is a positive real constant .

We note by $\Phi(k, k_0)$ the transition matrix of the linear part of the discrete system (4):

$$\Phi(k, k_0) = F_1^{k-k_0} \tag{6}$$

Let c and α are positive real constant and $\alpha \in]0,1[$.

By expressing the norm of the transition matrix Φ , we have:

$$\Phi(k, k_0) \leq c.\alpha^{k-k_0} \qquad \forall k \geq k_0 \tag{7}$$

In that case the solution $X(k)$ of the system (4) verifies the following inequality:

$$\|X(k)\| \leq c(\alpha + c\beta)^{k-k_0} \|X(k_0)\| \tag{8}$$

Then, the system (4) is exponentially stable if $\beta < 1 - \frac{\alpha}{c}$ and the stability is guaranteed in the domain defined by the following theorem [15]:

Theorem 1: Consider the discrete system (1) satisfying the fact that all eigenvalues are less than one and let c and α the positive numbers verifying $\alpha \in \left]0,1\right[$:

$$\left\|F_1^{k-k_0}\right\| \leq c.\alpha^{k-k_0} \forall k \geq k_0 \tag{9}$$

Then this system is asymptotically stable in the domain Ω_0 defined in (3) where R_0 is the unique positive solution of the following equation:

$$\sum_{k=2}^{q} c^{k-1} \|F_k\|.R_0^{k-1} - \frac{1-\alpha}{c} = 0 \tag{10}$$

Furthermore the stability is exponentially.
(See proof in [15])

4. The studied method of stability domain determination

We consider the initial stability domain Ω_0 characterized in theorem 1. Γ_0 is its boundary. By considering X_0^i a point of Ω_0 and X_k^i its image by the $F(.)$ function characterizing the system (1)

$$X_k^i = F^k(X_0^i) \tag{11}$$

It is obvious that X_k^i is belonging to the stability domain Ω_0 , and then one has:

$$\left\|X_k^i\right\| < R_0, \left\|F^k(X_0^i)\right\| < R_0 \tag{12}$$

In order to expand the GSR, we will search for a radius $r_{0,i}$ such as for every initial point X_0 checking:

$$\left\|X_0 - X_0^i\right\| \leq r_{0,i} \tag{13}$$

We obtain

$$X_k = F^k(X_0) \in \Omega_0 \tag{14}$$

Starting from the fact that in k iterations the state of the system attends the guaranteed domain of stability provides that X_0 is belonging in the stability domain.

Consider the following discard state:

$$\delta X_0 = X_0 - X_0^i \tag{15}$$

For $k \geq 1$

$$\delta X_k = X_k - X_k^i = F^k(X_0) - F^k(X_0^i) \tag{16}$$

By expressing $F^k(X_0)$ and $F^k(X_0^i)$ by their equations, δX_k can be written in terms of δX_0 as a polynomial function of degree $s = q^k$ where q is the degree of the $F(.)$ polynomial characterizing the system:

$$\delta X_k = E_1 \delta X_0 + E_2 \delta X_0^{[2]} + \cdots + E_s \delta X_0^{[s]}; s = q^k \tag{17}$$

$E_1, E_2, \cdots, E_s$ are matrices depending on k and X_0^i and they can easily expressed in terms of X_0^i and F_i.

The studied method of stability domain determination can be stated by the following theorem [16]:

Theorem 2: Let the following polynomial discrete system described by:

$$X_{k+1} = F(X_k) = F_1 X_k + F_2 X_k^{[2]} + \ldots + F_q X_k^{[q]} \tag{18}$$

and let Ω_0 the initial domain of stability of radius R_0 given by theorem 1 and Γ_0 its boundary, then for any point $X_0^i \in \Gamma_0$, the ball Ω_i centered on X_0^i and of radius $r_{0,i}$ the unique positive solution of the equation:

$$\|E_1\| r_{0,i} + \|E_2\| r_{0,i}^2 + \ldots + \|E_3\| r_{0,i}^s = R_0 - \left\|F^k(X_0^i)\right\| \tag{19}$$

is also a domain of stability of the considered system.

(See proof in [4])

5. Proposed improvement of the studied method of stability domain determination

By using the same principle of the method proposed above, we can characterize a new stability domain larger than the initial domain obtained by application of theorem 1.

Indeed, we find that the more iteration we make, the farther we move away from the equilibrium point. Based on this idea we propose an improved algorithm to obtain a large stability domain around the equilibrium point. The structure of the improved algorithm is as follows:

Step 1: Consider the studied discrete nonlinear system and determine the matrices $F_1, F_2, \ldots, F_q$ characterizing the polynomial model of the system. Determine the equilibrium points of the system by solving the equation $F(X_k) = 0$ and choose the interesting equilibrium state.

Step 2: Apply theorem 1 to determine a stability radius R_0 around the considered equilibrium state. Let Γ_0 be the boundary of the guaranteed stability ball Ω_0. Let N an integer $N \geq 2$ and put $j = 0$.

Step 3: Apply theorem 2 for one iteration $k = 1$ and for different values of $i \, (i = 1, \ldots, I)$, to enlarge the initial stability domain Ω_i. Let $r_{0,i}, i = 1, \ldots, I$ the different obtained radius of the stability balls centered on the boundary Γ_j of the stability ball Ω_j. Put $R_{j+1} = R_j + \min(r_{0,i})$ and Ω_{j+1} the ball of radius R_{j+1} centered on the equilibrium state.

Step 4: Put $j \leftarrow j + 1$ and repeat step 3 while $j \leq N$.

This improved algorithm allows to obtain an enlarged stability domain of the considered discrete nonlinear system. This improvement is illustrated by the simulation study of the following section.

6. Simulation Study

To show the efficiency of the improved algorithm proposed in the last section, we consider its implementation for the stability domain estimation and enlargement of an l-link manipulator system operated by a DC motor [11], described by the following continuous state representation:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\phi_1 x_1 - \phi_2 x_2 + \phi_3 x_3 + \dfrac{1}{6}\phi_1 x_1^3 \\ \dot{x}_3 = -\phi_4 x_2 - \phi_5 x_3 \end{cases} \tag{20}$$

where $\phi_1 = \dfrac{m.g.l}{M}, \phi_2 = \dfrac{D}{M}, \phi_3 = \dfrac{k_i}{M}, \phi_4 = \dfrac{k_{m3}}{k_{m1}}, \phi_5 = \dfrac{k_{m2}}{k_{m1}}$.

m represents the mass of the manipulator, g is the gravity, l is the distance of the center of mass of the link from the joint, D is the damping coefficient and $k_{mi}, i = 1,2,3$ are positive constants in electrical subsystem.

The nonlinear system can be expressed by the third degree polynomial system:

$$\dot{X} = A_1 X + A_2 X^{[2]} + A_3 X^{[3]} \tag{21}$$

with $A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -\phi_1 & -\phi_2 & \phi_3 \\ 0 & -\phi_4 & -\phi_5 \end{bmatrix}, A_2 = [0_{3\times 9}], A_3 = \begin{bmatrix} \frac{1}{6}\phi_1 & \\ 0 & 0_{3\times 26} \\ 0 & \end{bmatrix}$

The discretization of the state equation (21) using Newton-Raphson technique with the sampling period T leads to the following discrete state equation of the l-link manipulator [12]:

$$X_{k+1} = F_1 X_k + F_2 X_k^{[2]} + F_3 X_k^{[3]} \tag{22}$$

with:

$$F1 = \begin{bmatrix} 1 & T & 0 \\ -\phi_1 T & 1-\phi_2 T & \phi_3 T \\ 0 & -\phi_4 T & 1-\phi_5 T \end{bmatrix}, F_2 = [0_{3\times 9}], F_3 = \begin{bmatrix} \frac{1}{6}\phi_1 T & \\ 0 & 0_{3\times 26} \\ 0 & \end{bmatrix}$$

According to the following parameters:

$\phi_1 = 1.47, \phi_2 = 1, \phi_3 = 1, \phi_4 = 200, \phi_5 = 10, T = 0.01s$.

One obtains the numerical values of the matrices F_1, F_2, F_3

$$F1 = \begin{bmatrix} 1 & 0.01 & 0 \\ -0.014 & 0.99 & 0.01 \\ 0 & -2 & 0.9 \end{bmatrix}, \quad F_2 = [0_{3\times 9}], \quad F_3 = \begin{bmatrix} 0.0024 & \\ 0 & 0_{3\times 26} \\ 0 & \end{bmatrix}$$

Equation (22) has a linear asymptotically stable matrix F_1, which verifies the inequality (9) with $c = 7.73$ and $\alpha = 0.99$. We can conclude that the origin is exponentially stable for each initial state X_0 included in a ball Ω_0 centered in the origin and of radius $R_0 = 0.0245$.

Ω_0 could be considered as an initial guaranteed stability region and it is shown in figure 1.
In figure 2, we show the balls domains obtained by applying 450 iterations of the proposed method algorithm.
By running the improved algorithm, we can enlarge the initial stability domain until obtaining a radius $R = 0.1334$ for 450 iterations ($R_{j=450} = 0.1334$).

Figure 3 shows the difference between the initial domain and the largest domain obtained by applying the developed improvement.

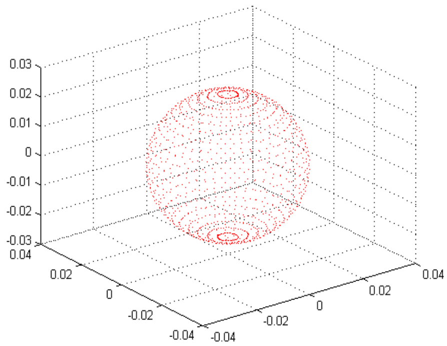

Figure 1: Guaranteed stability domain of system (22)

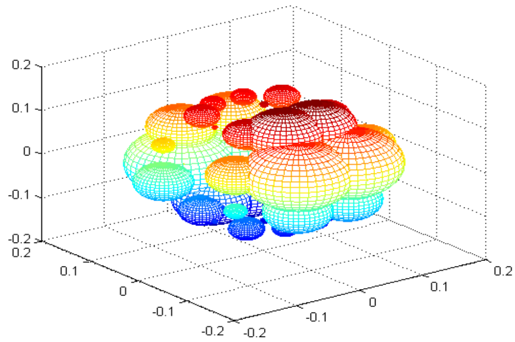

Figure 2: New stability domain obtained by applying 450 iterations of the improved algorithm

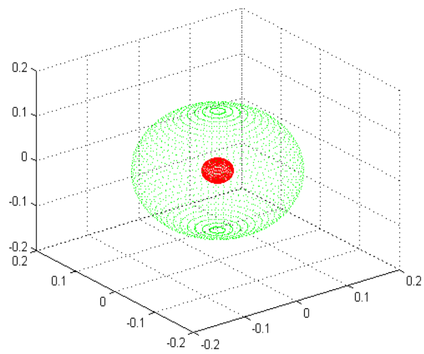

Figure 3: Enlargement of the initial stability domain of system (22) to the final area obtained by the improved algorithm

7. CONCLUSION

We have presented in this work new tools for the analysis of stability discrete dynamical systems. We have proposed a specific framework for estimating and expanding the domain of attraction of a class of nonlinear discrete polynomial systems.
Since the domain of attraction is a compact state space, we developed an improvement of a new approach that will enlarge a region of asymptotic stability around a stable equilibrium point. Our development is based on a non-Lyapunov method, we have prepared a rigorous analytical development for a significant expansion of an initial domain of attraction relatively reduced.

The implementation of the developed method has been illustrated in the case of discrete nonlinear system example, where it has been shown that one can achieve significantly an enlarged domain of asymptotic stability.

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