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Design of a fractional order PID controller (FOPID) for a class of fractional order MIMO systems

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Abstract- This paper deals with the design of a Fractional-order proportional-integral-derivative (FOPID) by using the different approximation methods, namely: Oustaloup, Matsuda, Carlson, for a class of MIMO fractional-order systems. A comparative study between the FOPID implemented via the approximation methods above cited and the classical PID is carried out. In addition, the results of a numerical simulation of a synchronous machine are presented. From this synthesis of controllers, we conclude that FOPID controllers are more efficient than PID in terms of degree of stability.

Keywords: MIMO systems, fractional-order systems, PID controllers, fractional-order PID, approximation of the fractional operators.

1. INTRODUCTION

Fractional-order calculus is an area of mathematics that deals with derivatives and integrals from non-integer orders. In other words, it is a generalization of the traditional calculus that leads to similar concepts and tools, but with a much wider applicability. In the last two decades, fractional calculus has been rediscovered by scientists and engineers and applied in an increasing number of fields, namely in the area of control theory. The success of fractional-order controllers is unquestionable with a lot of success due to emerging of effective methods in differentiation and integration of non-integer order equations [1]-[14].

Fractional-order proportional-integral-derivative (FOPID) controllers have received a considerable attention in the last years both from academic and industrial point of view. In fact, in principle, they provide more flexibility in the controller design, with respect to the standard PID controllers, because they have five parameters to select (instead of three). However, this also implies that the tuning of the controller can be much more complex. In order to address this problem, different methods for the design of a FOPID controller have been proposed in the literature. However, the implementation of a FOPID has been generally made via an appropriate approximations of those fractional-orders, namely Oustaloup, Matsuda, Carlson, or via the so-called exact analytical formula (i.e. without any approximation) [15]-[22].

This paper deals with the design of a FOPID by using the different approximation methods above cited, for a class of MIMO fractional-order systems. A comparative study between the FOPID (by using the different approximation methods) and the classical PID will be carried out.

This paper is organized as follows. Section 2 includes basic concepts in fractional calculus and the different used approximations for FOPID implementation. In Section 3 the Bode's

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ideal loop (BIL) with temporal and frequency analysis is presented. In section 4 the determination of the representation of state of such a non-integer system is presented and the fractional order multiple input and multiple-output (MIMO) systems models to be controlled are studied in the state representation. The design of FOPID is detailed in section 5. The application of FOPID, made according to the approximations of: Oustaloup, Matsuda, Carlson and the exact analytical formula (without any approximation) and a comparison of the corresponding results using fractional MIMO system are presented. Finally, concluding remarks are drawn in Section 7.

2. FRACTIONAL ORDER OPERATORS

The fractional order operation (derivator case) is given by:

$$y(t) = \tau^n D^n u(t) \tag{1}$$

τ indicates the differentiation time constant and $n \in C$ the complex order of the derivation (Re(n)=α which can be higher or lower than 0, the operator considered being then either a derivator, or an integrator).

Using the Laplace's transform and under null initial conditions, we can write (1) as follows:

$$Y(s) = (\tau s)^n U(s) \tag{2}$$

Posing now $\omega_u = 1/\tau$, called transitional frequency, we get:

Thus, the transmittance is given by:

$$D(s) = \left(\frac{s}{\omega_u}\right)^n \tag{3}$$

The calculation of fractional derivatives and integrals of an order α (Re(n)=α, see equations 1-3) for a specified function is generally very difficult by using the analytical methods. By consequence, a numerical approximation is necessary [3][6][7][12]-[14]. We use a function fotf (fractional order transfer function) of the Matlab toolbox to determine the so-called exact responses of fractional order derivative or integral and to compare them with the various following approximations:

2.1 Oustaloup's approximation

The method [7] is based on the approximation of a function of the form:

$$T(s) = s^q, \;\; q \in R^+ , \tag{4}$$

By a rational function:

$$\hat{T}(s) = C \prod_{k=-N}^{N} \left(\frac{1+\frac{s}{w_k}}{1+\frac{s}{w'_k}}\right) \tag{5}$$

Where the parameters of this function can be determined via the following formulas :

$$w_0' = \alpha^{-0.5} w_u;\ w_0 = \alpha^{0.5} w_u;\ \frac{w_{k+1}'}{w_k'} = \frac{w_{k+1}}{w_k} = \alpha\eta > 1;\ \frac{w_{k+1}'}{w_k} > 0;\ \frac{w_k}{w_k'} = \alpha > 0;\ N = \frac{log(\frac{w_N}{w_0})}{log(\alpha\eta)};\ q = \frac{log(\alpha)}{log(\alpha\eta)} \quad (6)$$

With w_u being the frequency of the unit gain and the center frequency of a band of the frequencies geometrically distributed around it, i.e. : $w_u = \sqrt{w_h w_b}$ where w_h and w_b are respectively the high and low transient-frequencies.

2.2 Matsuda's approximation

The method suggested by [8] is based on the approximation of the fractional-order operator $\mathrm{T(s)} = \mathrm{s}^{\alpha}$ by a rational function identified by using its gain. The gain is calculated by using M frequencies left again in a waveband $[\mathrm{w_0}, \mathrm{w_M}]$ in which the approximation is made. For a set of selected points $\mathrm{w_i}, \mathrm{i} = 0,1,2, \dots, \mathrm{M}$, the approximation takes the following form:

$$\hat{T}(s) = a_0 + \frac{s - w_0}{a_1 + \frac{s - w_1}{a_2 + \frac{s - w_2}{a_3 + \cdots}}} \quad (7)$$

$$a_i = v_i(w_i);\ v_0(w) = |G(jw)|;\ v_{i+1}(s) = \frac{s - w_i}{v_i(s) - a_i};\ For: \quad i = 0,1,2, \dots M \quad (8)$$

The approximated model is obtained by replacing each fractional operator of the irrational explicit transfer function by its approximation.

2.3 Carlson's approximation

The method suggested by Carlson in [9], derived from a regular Newton process used for the iterative approximation of **α** roots, can be considered as a membership of this group. The starting point of this method is the report of the following relations:

$$(H(s))^{1/\alpha} - (G(s)) = 0; \quad H(s) = s^{\alpha} \quad (9)$$

Defining $\alpha=1/q$, $m=q/2$, in each iteration, starting from the initial value$H_0(s) = 1$, an approximated rational function is obtained in the form:

$$H_i(s) = H_{i-1}(s) \frac{(q-m)(H_{i-1}(s))^2 + (q+m)G(s)}{(q+m)(H_{i-1}(s))^2 + (q-m)G(s)} \quad (10)$$

3. BODE'S IDEAL LOOP (BIL)

The Bode's ideal loop (BIL) transfer function was proposed for the first time by Bode in its work on the design of the amplifiers with feedback in 1945 [6][10]. The diagram of such an amplifier with unity-gain feedback is given by the figure (1), whose transfer function in open loop is defined by an integrator of a fractional nature of the form:

$$T(s) = \frac{A}{s^{\alpha}} \ , \ 1 < \alpha < 2 \quad (11)$$

Where α is the slope of the ideal characteristic of the gain.

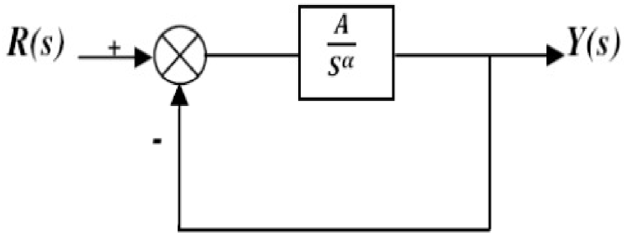

Figure 1 Diagram of Bode's ideal loop (BIL)

3.1 Temporal analysis of the BIL

The transfer function in closed-loop of BIL has the following form:

$$H(s) = \frac{Y(s)}{R(s)} = \frac{A}{S^{\alpha}+A} \tag{12}$$

Its step response is given by :

$$y(t) = At^{\alpha}E_{\alpha,\alpha+1}(-At^{\alpha}) \tag{13}$$

Where $\mathbf{E_{\alpha,\alpha+1}}$ is the function of Mittag-Leffler:

$$E_{\alpha,\alpha+1}(\theta) = \sum_{k=0}^{\infty} \frac{\theta^k}{\Gamma(1+k.\alpha)} \tag{14}$$

The step responses of the system $\mathbf{H(s)}$ are characterized by a damping coefficient $\boldsymbol{\delta}$, a natural pulsation $\boldsymbol{\omega_n}$ and an eigen frequency $\boldsymbol{\omega_p}$ given by the following formulas:

$$\delta = -\cos\left(\frac{\pi}{\alpha}\right) \quad ; \quad \omega_n = A^{\frac{1}{\alpha}} \quad ; \quad \omega_p = A^{\frac{1}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right) \tag{15}$$

The maximum overshoot can be expressed according to the order α by [12]:

$$M_p = \frac{h_{max}-h(\infty)}{h(\infty)} \approx 0.8(\alpha-1)(\alpha-0.75)\; ;\; 1 < \alpha < 2 \tag{16}$$

The time of the first overshoot and the boarding time (2% to 90%) can be given in an approximate way by the following expressions:

$$\omega_u t_d \approx \frac{1.106(\alpha-0.255)^2}{(\alpha-0.921)^2};\; 1 < \alpha < 2 \tag{17}$$

and $$\omega_u t_m = \frac{0.131(\alpha+1.157)^2}{(\alpha-0.724)};\; 1 < \alpha < 2 \tag{18}$$

Where ω_u is the transitional frequency. The figure (2) represents the step responses of the system (12) for $\alpha = 1.5$ and for the different values of A. This figure shows that the maximum overshoot is approximately of 29.62% and independent of the gain A. This property is called iso-damping (ξ = constante $\approx$ 0.35).

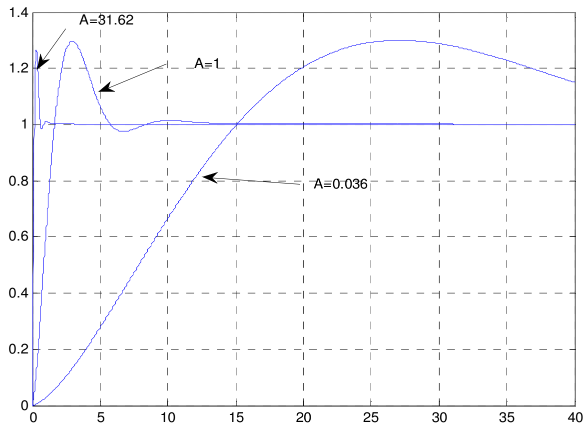

Figure 2 The step responses of the BIL for $\alpha = 1.5$ and various values of A

3.2 Frequential analysis of the BIL

The Bode's diagram of the direct chain of the ideal loop of Bode is given by the fig. 3. The frequential response is characterized by a slope of $-20\alpha dB/dec$ and a constant phase of $-\frac{\alpha\pi}{2}$ rad. Thus, the phase margin in closed-loop is independent of the gain A and equal to: $\Phi_m = (1 - \frac{\alpha}{2})\pi$.

The factor of resonance M_r and the frequency of resonance ω_r can be determined in the same way like in the case of integer-order systems. Let $H(j\omega)$ be the transmittance in closed-loop:

$$H(j\omega) = \frac{A}{(j\omega)^\alpha + A} = \frac{A}{A + \omega^\alpha cos\left(\alpha\frac{\pi}{2}\right) + j\omega^\alpha sin\left(\alpha\frac{\pi}{2}\right)} \qquad (19)$$

Its module is given by:

$$|H(j\omega)| = \frac{A}{\sqrt{\omega^{2\alpha} + 2A\omega^\alpha cos\left(\alpha\frac{\pi}{2}\right) + A^2}} \qquad (20)$$

The module (21) has a maximum for:

$$\omega_r = \left(-Acos\left(\alpha\frac{\pi}{2}\right)\right)^{\frac{1}{\alpha}}\ ,\ \alpha > 1 \qquad (21)$$

Corresponding to a factor of resonance given by:

$$M_r = \frac{1}{sin\left(\alpha\frac{\pi}{2}\right)} \qquad (22)$$

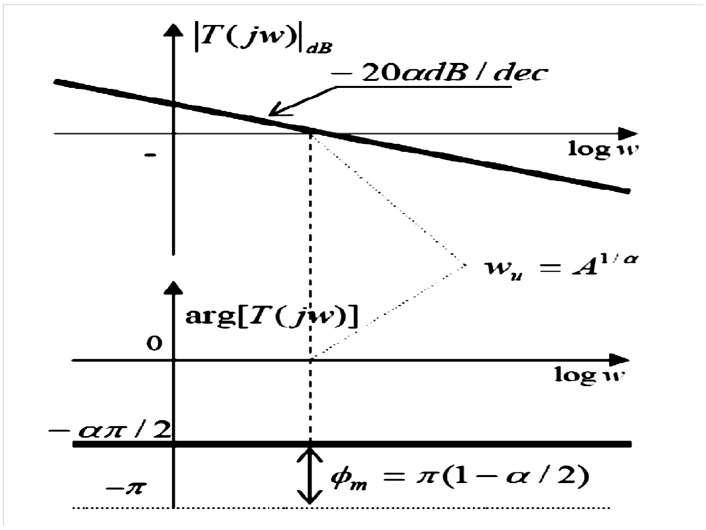

Figure 3 Bode's diagram in open-loop of BIL

4. FRACTIONAL–ORDER MIMO SYSTEMS

Fractional-order (FO) systems have attracted increasing interests, mainly due to the fact that many real-world physical systems are better characterized by FO differential equation. From the differential equations of a non-integer-order system, it is possible to build a state model using the fractional derivative of the system's state variables. In order to determine the state representation of such a non-integer –order system, let's consider the following differential equation obtained by linearization [1]-[14][19][20][23]:

$$\sum_{n=0}^{N} a_n \left(\frac{d}{dt}\right)^{\alpha_n} y(t) = \sum_{m=0}^{M} b_m \left(\frac{d}{dt}\right)^{\beta_m} u(t) \text{ où } \alpha_n, \beta_m \in R,\ \alpha_N \geq \beta_M \quad (23)$$

Where the vector α_n represents the vector of the derivative orders of the output *y(t)* classified in the ascending order. The vector β_n represents the vector of the derivative orders of the input *u(t).* The coefficients a_n and b_m are defined according to the parameters of the considered system.

To build the system of generalized states, one can carry out the following variables change :

- The first introduced variable, x_1, is defined like the derivative of order α_0 of *y(t);*
- The second variable x_2 is defined like the derivative of order α_1 of the output *y(t);*
- The i[th] variable is defined like the derivative of order α_i of the output *y(t).*

In general case, $\alpha_0 = 0$; the variable x_1 corresponds thus to the function $y(t)$.

The new variables and the relations between them are given by :

$$\begin{cases} \left(\frac{d}{dt}\right)^{(\alpha_0)} y(t) = x_1(t) \\ \left(\frac{d}{dt}\right)^{(\alpha_1)} y(t) = \left(\frac{d}{dt}\right)^{(\alpha_1-\alpha_0+\alpha_0)} y(t) = {x_1}^{(\alpha_1-\alpha_0)}(t) = x_2(t) \\ \vdots \\ \left(\frac{d}{dt}\right)^{(\alpha_{i-1})} y(t) = \left(\frac{d}{dt}\right)^{(\alpha_{i-1}-\alpha_{i-2}+\alpha_{i-2})} y(t) = {x_{i-1}}^{(\alpha_{i-1}-\alpha_{i-2})}(t) \\ \qquad = x_i(t) \\ \vdots \\ \left(\frac{d}{dt}\right)^{(\alpha_{N-1})} y(t) = \left(\frac{d}{dt}\right)^{(\alpha_{N-1}-\alpha_{N-2}+\alpha_{N-2})} y(t) = {x_{N-1}}^{(\alpha_{N-1}-\alpha_{N-2})}(t) \\ \qquad = x_N(t) \\ \left(\frac{d}{dt}\right)^{(\alpha_N)} y(t) = \left(\frac{d}{dt}\right)^{(\alpha_N-\alpha_{N-1}+\alpha_{N-1})} y(t) = {x_N}^{(\alpha_N-\alpha_{N-1})}(t) \end{cases} \quad (24)$$

Taking into account of this variables change, the equation (24) becomes:

$${x_N}^{(\alpha_N-\alpha_{N-1})}(t) = -\frac{a_0}{a_N} x_1(t) - \frac{a_1}{a_N} x_2(t) - \cdots - \frac{a_{N-1}}{a_N} x_N(t) + \sum_{m=1}^{M} \frac{b_m}{a_N} \left(\frac{d}{dt}\right)^{\beta_M} u(t) \quad (25)$$

From (25) and (26), we can write the state model as follows:

$$\begin{bmatrix} {x_1}^{(\alpha_1-\alpha_0)}(t) \\ {x_2}^{(\alpha_2-\alpha_1)}(t) \\ \vdots \\ {x_{N-1}}^{(\alpha_{N-1}-\alpha_{N-2})}(t) \\ {x_N}^{(\alpha_N-\alpha_{N-1})}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\frac{a_0}{a_N} & -\frac{a_1}{a_N} & \cdots & \cdots & -\frac{a_{N-1}}{a_N} \end{bmatrix} . \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{N-1}(t) \\ x_N(t) \end{bmatrix} + \begin{bmatrix} 0 \cdots 0 \\ 0 \cdots 0 \\ 0 \cdots 0 \\ 0 \cdots 0 \\ 0 \cdots 0 \\ \frac{b_1}{a_N} \cdots \frac{b_M}{a_N} \end{bmatrix} . \begin{bmatrix} u^{(\beta_1)}(t) \\ \vdots \\ u^{(\beta\ )}(t) \end{bmatrix} \quad (26)$$

Thus, the system (27) can be generally written in the form:

$$\begin{cases} \bar{x}^{(\bar{n})}(t) = A.\bar{x}(t) + B.\bar{u}(t) \\ \bar{y}(t) = C.\bar{x}(t) + D.\bar{u}(t) \end{cases} \quad (27)$$

Thus, as in the integer-order case, a representation of non-integer-order state comprises two equations; an equation of state generalized in which the vector of state is not any more the object of a unit derivation but of a derivation of non-integer real order and an equation of observation.

We also see appearing the vectorial operator $(\bar{n}) = (\alpha_1 - \alpha_0, \ldots, \alpha_N - \alpha_{N-1})$ steady to the vector of state. Thus, this vector is not inevitably composed of identical terms. Nevertheless, in the studies of stability, we will systematically seek an arrangement of the state variables so that the components of this vector (n) are the same ones. It could then be comparable with a real number n [14][15][20][23].

5. DESIGN OF A FRACTIONAL ORDER PID CONTROLLER

Recently, FOPID controller design has been increasingly used in the control area by more and more researchers. Compared with the classical PID controller, FOPID controller have the potential to improve the control performance, because extra real parameters α and β are involved. However, to the best of our knowledge, there are two common assumptions for the considered plant, which are (i) The plant can be modeled as first or second-order systems, even plus a time delay item. (ii) The plant is of the form of single input single output system. And few results have been obtained for multivariable FOPID controller design for the FO systems, especially when the parameters of the FO system are interval uncertainties. Therefore, it is highly make sense that developing methods to determine the parameters of the FOPID controllers for FO multi-inputs multi-outputs high order systems. The above all motivate this present study.

Podlubny proposed a generalization of the classical PID controller, called fractional-order PID controller defined by the following transfer function [15]:

$$C(S) = \frac{U(S)}{E(S)} = K_p + \frac{K_i}{S^\alpha} + K_d S^\beta \ , \ \ \alpha, \beta \in R^+ \tag{28}$$

Several methods were proposed for the design of this type of order [11][15]-[18][21][23]. Moved by the remarkable performances characteristic in quality of robustness of the ideal loop of Bode. In what follows, we will propose, the design of a control $\mathrm{PI^\alpha D^\beta}$ which ensures the same frequential and temporal behavior that found while being based on the ideal loop of Bode in closed loop.

After having fixed the fractional orders α and β starting from the behavior frequential of the system of open loop control, which must be equivalent to that of the ideal transfer function of Bode. We established a simple design of a control system of a fractional nature elementary based on the parameters m and ω_u. Consequently, in this section we use the fractional integrator of order of the equation (12) like a function of reference $G_m(s)$ for the control device $PI^\alpha D^\beta$ [15].

We start by considering the system in closed loop shown in the figure (4), where C(s) is the controller $PI^\alpha D^\beta$ and $G_p(s)$ is the transfer function of the process, characterized by an asymptotic order at low frequency $0 \le n' \le 2$ and high frequency $2 \le n \le 4$ with $n' < n$ (fig. 4).

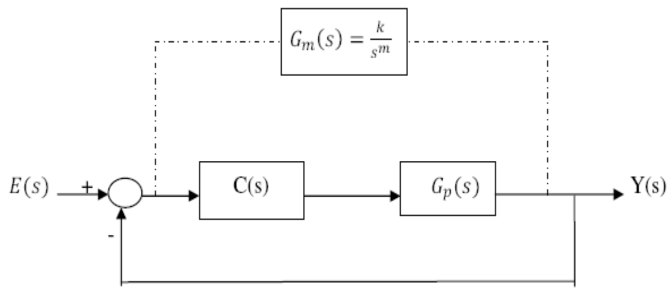

Figure 4 Closed loop system control

The ideal transfer function of the controller $PI^\alpha D^\beta$ is the following form:

$$C(s) = k_p\left(1 + \frac{T_i}{s^\alpha} + T_d s^\beta\right) \ , \qquad k_p, T_i \ et\ T_d \ \in R^+ \tag{29}$$

With α and β are positive real numbers, k_p is the proportional gain, T_i is a constant of integration and T_d is a constant of derivation.

The cut-off frequency ω_u is considered to be higher 10 times than the transitional frequency of the process.

The transfer function of the model of reference in open loop has the form:

$$G_m(s) = \frac{1}{\left(\frac{s}{\omega_u}\right)^m} \quad 1 < m < 2 \ \ et \ \ \omega_u \in R^+ \tag{30}$$

Where ω_u and m are fixed according to the performances desired in closed loop.

The method of synthesis of $\mathrm{PI^\alpha D^\beta}$ based on the interpretation of the transfer function in open loop T(s) which can be written in the form:

$$T(s) = C(s)G_p(s) \tag{31}$$

Transmittance T(s) can be here regarded as the approximation of the transfer function in open loop of the model of reference $G_m(s)$, and then we can write:

$$k_p\left(1 + \frac{T_i}{s^\alpha} + T_d s^\beta\right)G_p(s) \approx \frac{k_u}{s^m} \; ; \quad k_u = \omega_u^m \tag{32}$$

We give the waveband: $\omega_{min} \ll 10\omega_0, \omega_u \ll \omega_{max}$
Where ω_0 is the cut-off frequency of the process, transmittance T(s) should present:

- An asymptotic slope of $-20dB/dec$ with low and the high frequency of the limited bandwidth $[\omega_{min}, \omega_{max}]$ which makes it possible to calculate the parameters α and β.
- A same gain with the reference $G_m(s)$ into high and low frequencies, thus, the initial values of the parameters $T_i{}'$ and $T_d{}'$ can be estimated with an initial value of k_p considered $k_p = 1$.
- A cut-off frequency equal ω_u, the parameter k_p can be deduced by using the initial values $T_i{}'$ and $T_d{}'$.
- Finally, we make the adjustment of the parameters $T_i{}'$ and $T_d{}'$ to obtain the parameters values: $T_i = k_p T_i{}'$ And $T_d = k_p T_d{}'$.

Taking as n and n' an asymptotic order of the process $G_p(s)$ at high and low frequencies respectively, the parameters of the controller can be given by the following equations [15]:

$$\beta = n - m \; , \quad \alpha = m - n' \tag{33}$$

$$T_d' = \frac{K_u}{|G_p(j\omega_{max})|\omega_{max}^n} \tag{34}$$

$$T_i' = \frac{K_u}{|G_p(j\omega_{min})|\omega_{min}^{n'}} \tag{35}$$

$$k_p = \frac{|G_p(j\omega_u)|^{-1}}{|1+T_i'(j\omega_u)^{-\alpha}+T_d'(j\omega_u)^\beta|} \tag{36}$$

$$T_i = k_p T_i{}' \quad \text{and} \quad T_d = k_p T_d{}' \tag{37}$$

6. SIMULATION RESULTS

In this section, we have focused on synchronous machine modeling (The synchronous machine used here is a Darlington type). It is then necessary not only to supply accurate models of machines over a wide range of frequencies, but also to include knowledge in models. Moreover, model order has to be reduced as robust control techniques lead also to high-order controllers [23]. The system order is then so high that classical dynamic studies like stability become difficult. In most generators, as frequency increases, induced currents can no longer be neglected. Equivalent circuits of synchronous machines are then improved by including ladder elements with constant parameters [24][25]. However, as this effect is a distributed phenomenon described by partial differential equations, classical improved equivalent circuits must include in theory an infinite number of lumped and constant parameters. Finally, half-order linear impedances can be included in the Park equivalent circuits of synchronous generators (Fig. 5) following some physical conditions [24]-[26]. Then, for d-axis modeling, no induced currents expand in armature windings or field windings for low frequencies; they are then modeled by constant parameters.

After modeling synchronous machine we will stabilize their currents id and iq using both PID and FOPID (with different approximation methods).

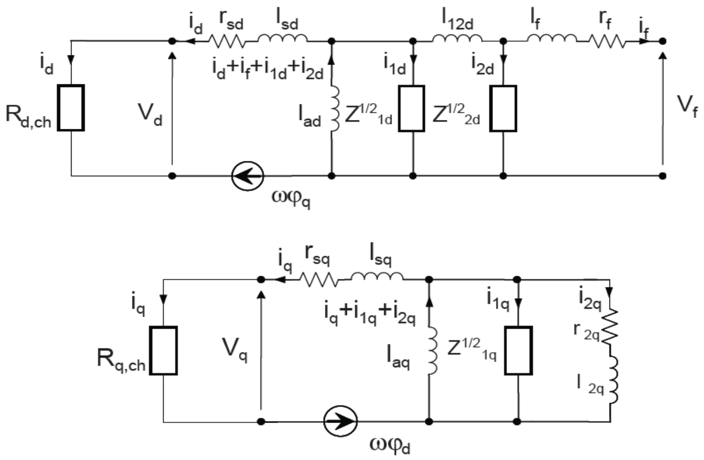

Figure 5 Non integer order d and q-axis equivalent circuits of a synchronous machine

The system of equations governing the electric circuits is given in the field of Laplace by the flowing system equations (38):

$$\begin{cases} R_{d,ch}.i_d = -r_{sd}.i_d - l_{sd}.s.i_d - l_{ad}.s.\left(i_d + i_f + i_{1d} + i_{2d}\right) \\ V_f = -r_f.i_f - l_f.s.i_f - l_{ad}.s.\left(i_d + i_f + i_{1d} + i_{2d}\right) \\ \qquad -l_{12d}.s.\left(i_f + i_{2d}\right) \\ 0 = -R_{2d}.\left(1 + \left(\frac{s}{\omega_{2d}}\right)^{\frac{1}{2}}\right).l_{2d} - l_{ad}.s.\left(i_d + i_f + i_{1d} + i_{2d}\right) \\ \qquad - l_{12d}.s.\left(i_f + i_{2d}\right) \\ 0 = -\frac{s.L_{1d}.i_{1d}}{1+\left(\frac{s}{\omega_{1d}}\right)^{\frac{1}{2}}} - l_{ad}.s.\left(i_d + i_f + i_{1d} + i_{2d}\right) \\ R_{q,ch}.i_q = -r_{sq}.i_q - l_{sq}.s.i_q - l_{aq}.s.\left(i_q + i_{1q} + i_{2q}\right) \\ 0 = -r_{2q}.i_{2q} - l_{2q}.s.i_{2q} - l_{aq}.s.\left(i_q + i_{1q} + i_{2q}\right) \\ 0 = -s.L_{1q}.i_{1q} - \left(1 + \left(\frac{s}{\omega_{1q}}\right)^{\frac{1}{2}}\right).l_{aq}.s.\left(i_q + i_{1q} + i_{2q}\right) \end{cases} \tag{38}$$

A system of generalized state space is then built by using the following change of variables:

$$\begin{cases} x_1 = i_d\ ; \\ x_2 = x_1^{(1/2)} = i_d^{(1/2)}\ ; \\ x_3 = i_f\ ; \\ x_4 = x_3^{(1/2)} = i_f^{(1/2)}\ ; \\ x_5 = i_{2d}\ ; \\ x_6 = x_5^{(1/2)} = i_{2d}^{(1/2)}\ ; \end{cases} \qquad \begin{cases} x_7 = i_{1d}^{(1)}; \\ x_8 = i_q\ ; \\ x_9 = x_8^{(1/2)} = i_q^{(1/2)}; \\ x_{10} = i_{2q}\ ; \\ x_{11} = x_{10}^{(1/2)} = i_{2q}^{(1/2)}; \\ x_{12} = i_{1d}^{(1)}\ ; \end{cases} \tag{39}$$

We can write (38) as a following state space model:

$$\bar{x}^{\left(\frac{1}{2}\right)} = A.\bar{x} + B.\bar{u}\ ; \tag{40}$$

Where:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{2.1} & 0 & A_{2.3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{4.1} & 0 & A_{4.3} & 0 & 0 & A_{4.6} & A_{4.7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{6.1} & 0 & A_{6.3} & 0 & A_{6.5} & A_{6.6} & A_{6.7} & 0 & 0 & 0 & 0 & 0 \\ A_{7.1} & A_{7.2} & A_{7.3} & A_{7,4} & A_{7.5} & A_{7.6} & A_{7.7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{9,8} & 0 & A_{9,10} & 0 & A_{9,12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{11,8} & 0 & A_{11,10} & 0 & A_{11,12} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_{12,8} & A_{12,9} & A_{12,10} & A_{12,11} & A_{12,12} \end{bmatrix};\ B = \begin{bmatrix} 0 & 0 \\ k_2 & 0 \\ 0 & 0 \\ k_4 & 0 \\ 0 & 0 \\ k_6 & 0 \\ k_7 & k_7' \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix};\ u = \begin{bmatrix} V_f & V_f^{(1/2)} \end{bmatrix} \tag{41}$$

The 05 parameters of $PI^{\alpha}D^{\beta}$ will be calculated using the approximations of Oustaloup, Matsuda, Carlson and the exact analytical formulas to control and stabilize synchronous machine's currents i_d and i_q using d and q-axis in the waveband [10^{-3},10^{3}] rad/s. We have already chosed the order of approximation n = 4 in the simulation.

The figures (6-9) respectively represent the step responses and the Bode's diagrams of the studied system (synchronous machine currents i_d and i_q in the reference axis of Park) controlled by PID and FOPID (with various approximations) in open loop as illustrated in fig. 4.

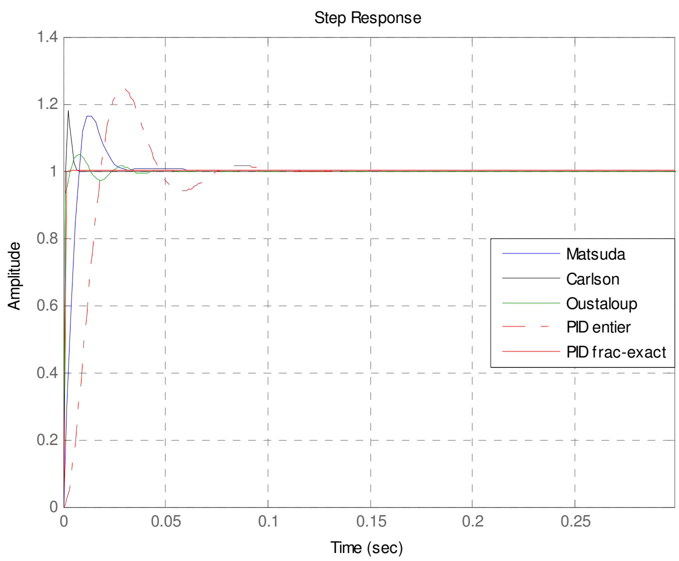

Figure 6 Step responses of the current i_d (state x_1) controlled by classical PID and FOPID with various approximations

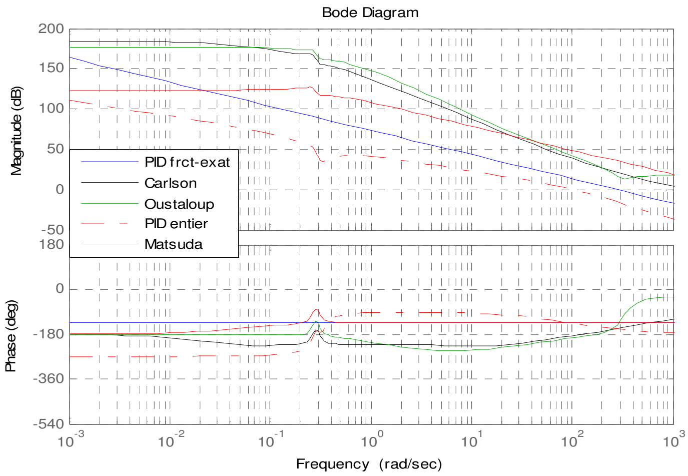

Figure 7 Bode's diagrams of the current i_d (state x_1) controlled by classical PID and FOPID with various approximations

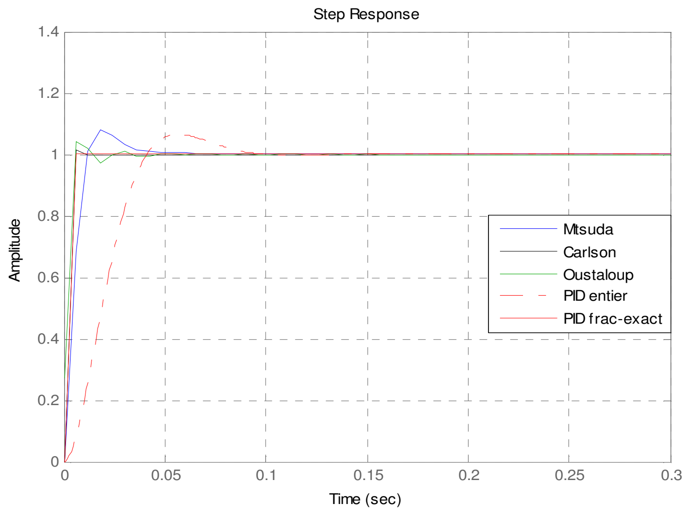

Figure 8 Step responses of the current i_q (the state x_8) controlled by classical PID and FOPID with various approximations

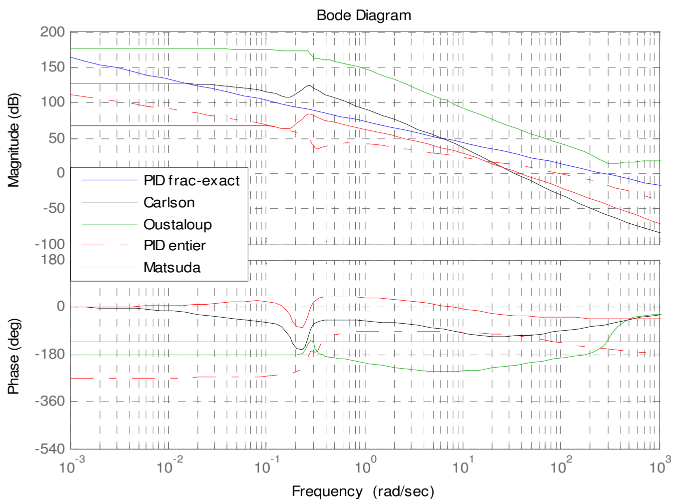

Figure 9 Bode's diagrams of the current i_q (state x_8) controlled by classical PID and FOPID with various approximations

Figures 6 and 8, show that the FOPID stabilize the currents and ensure the best results of robustness by comparing them with those of an integer order (a better speed).

The performances of PID and FOPID with various approximations are summarized in the tables below:

TABLE 1: PID and FOPID PERFORMANCES FOR CURRENT i_d

	T_m (s)	$T_{r5\%}$ (s)	D%	V_{finale}	MP(°)
FOPID (without approximation)	**0.002**	**0.003**	**00**	**01**	**45**
FOPID with Oustaloup approximation	**0.001**	**0.004**	**05**	**01**	**45**
FOPID with Matsuda approximation	**0.007**	**0.021**	**16**	**01**	**47.5**
FOPID with Carlson approximation	**0.001**	**0.003**	**18**	**01**	**50**
Classical PID	**0.017**	**0.062**	**25**	**01**	**43.8**

TABLE 2: PID and FOPID PERFORMANCES FOR CURRENT i_q

	T_m (s)	$T_{r5\%}$ (s)	D%	V_{finale}	MP(°)
FOPID (without approximation)	**0.005**	**0.007**	**00**	**01**	**45**
FOPID with Oustaloup approximation	**0.005**	**0.007**	**02**	**01**	**45**
FOPID with Matsuda approximation	**0.01**	**0.026**	**07**	**01**	**47.5**
FOPID with Carlson approximation	**0.005**	**0.007**	**01**	**01**	**50**
Classical PID	**0.03**	**0.054**	**08**	**01**	**43.8**

With regard to the comparison of the performances of robustness (Tab. 1 and Tab. 2), they reveal clearly that the robustness is much better in the case of the FOPID and its approximations, in favor of the approximation suggested by Oustaloup.

7. CONCLUSION

In this paper, we stabilize and control a fractional MIMO system (synchronous machine) using PID and FOPID with approximations (Oustaloup, Carlson and Matsuda). The simulation results show that the FOPID is efficient and more robust than a traditional PID in term of degree of stability. Therefore, we can control fractional system defined by its generalized state form by a FOPID starting from a simple loop of regulation, but the calculation of the FOPID five parameters will be very difficult in the case of the non-integer systems not given by its generalized state form.

REFERENCES

[1] K. B. Oldham, and J. Spanier, “The fractional calculus”, Academic Press, New York et Londres, 1974.
[2] K. S. Miller, and B. Ross, “An introduction to the fractional calculus and fractional differential equations”, John Wiley and Sons, 1993.
[3] S. G. Samko, A. A. Kilbas, and O. I. Marichev,” Fractional integrals and derivatives:Theory and applications”, Gordon and Breach, 1987.
[4] D. Matignon, and G. Montseny (dir.), “Fractional differential systems: Models,methods, and applications”, SIAM, décembre 1998 (disp. : http://www.emath.fr/Maths/Proc/Vol.5).
[5] G. Montseny, J. Audounet, and D. Matignon, ”Diffusive representation for pseudo-differentially damped non-linear systems”, dans Isidori A., Lamnabhi-Lagarrigue F., Respondek W. (dir.), Nonlinear control in the year 2000 (Paris, France), Springer-Verlag, vol. 2, pp. 163-182, 2000.
[6] A. Charef et al., “Fractional System as represented by singularity function”, IEEE Transaction on Automatic Control, vol. 37, n° 9, 1992.
[7] A. Oustaloup et al., ”Frequency-band complex non integer differentiator: characterization and synthesis”, IEEE transaction on Circuit and Systems, 2000.
[8] K. Matsuda and H. Fuji, “Optimized wave-absorbing control: Analytical and experimental results”, Journal of Guidance, Control, and Dynamics, 16, No. 6, 1146-1153, 1993.
[9] G. E. Carlson and C. A. Halijak, “Approximation of fractional capacitors (1/s)1/n by a regular Newton process”, IRE Transactions on Circuit Theory, 1964.
[10] H. W. Bode, Network Analysis and Feedback Amplifier Design, Van Nostrand. New York, 1945.
[11] R.S. Barbosa, J. A Tenreiro Machado. and I. M. Ferreira, “A Fractional Calculus a Perspective of PID Tuning”, Proceedings of DETC. 03ASME. 2003.
[12] Y. Q. Chen, I. Petras and D. Xue, “Fractional Order Control” - A Tutorial”, American Control Conference, 2009.
[13] C. A. Monje, Y. Q. Chen, B. M. Vinagre, D. Xue and V. Feliu, Fractional-Order Systems and Controls: Fundamentals and Applications. Springer, Heidelberg (2010).
[14] J. Sabatier and C. Farges, “On stability of fractional order systems”, International Journal of Bifurcation and Chaos (2011, in press).
[15] I. Podlubny, “Fractional-order systems and $PI^{\alpha}D^{\beta}$controllers”, IEEE Transactions z on Automatic Control , 1999.
[16] I. Petráš, L. Dorčák and I. Koštial, “Control quality enhancement by fractional ordes controllers”, Acta Montanistica Slovaca Ročník 3, 2, pp. 143-148, 1998.
[17] A. Djouambi, A. Charef and T. Bouktir, “Fractal robustness and parameter tuning PIμDm controllers”, proceedings of the 5th int. conf. on signal, speech and image processing , Corfu, Greece, pp. 158-159. August 17-19, 2005.
[18] C. A.Monje, B.M. Vinagre, Chen Y. Q., and V. Feliu, “ Proposals for fractional PID-tuning”, In Proceedings of the first IFAC Symposium on Fractional differentiation and its Application (FDA04), Bordeaux, France, 2004.
[19] H. S. Ahn and Y. Chen, “Necessary and sufficient stability condition of fractional-order interval linear systems”, Automatica, vol. 44, pp. 2985–2988, 2008.
[20] H. S. Ahn, Y. Chen, and I. Podlubny, “Robust stability test of a class of linear time-invariant interval fractional-order system using Lyapunov inequality”, Appl. Math. Comput., vol. 187, pp. 27–34, 2007.
[21] D. Xue, C. Zhao, and Y. Q. Chen. Fractional order PID control a DC-motor with elatic shaft: A case study. In Proceedings of the 2006 American Control Conference, pages 3182–3187,Minnesota, USA, 2006.
[22] Y. Luo, Y. Q. Chen, C. Y. Wang, and Y. G. Pi. “Tuning fractional order proportional integral controllers for fractional order systems”, Journal of Process Control, vol. 20, pp. 823–831, 2010.
[23] S. Skogestad, I. Postlethwaite, Multivariable feedback control. Analysis and design. John Wiley &sons. 2nd edition. 2005.
[24] I. Kamwa, P. Viarouge, H. Le-Hui, J. Dickinson,”A frequency maximum likelihood of synchronous machine high order models using SSFR data”, IEEE Trans. on Energy Conversion, Vol. 7, No.4, pp. 525-536, 1992.

[25] I.M. Canay, “Causes of discrepancies on calculation of rotor quantities and exact equivalent diagrams of synchronous machine”, IEEE Trans. On Power Apparatus, Vol. 88, pp. 1114-1120, 1969.
[26] D. Riu, N. Retière, M. Ivanès, “Induced currents modelling by half-order systems. Application to hydro- and turbo-alternators”, IEEE Trans. On Energy Conversion, Vol. 18, pp. 94-99, 2003.