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Constrained Nonlinear Model Predictive Control of Hybrid Dynamic Systems

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In this paper, a nonlinear model predictive control law of hybrid dynamic system is presented. An optimal switching strategy is obtained by an exhaustive search. It is shown that the online optimization of a non convex optimal control problem is not required. In fact, by using a Taylor series expansion of the output tracking error, the solution of the optimal control law can arise from a convex quadratic programming problem which incorporates rigorously inputs constraints. The main feature of the proposed method is its ability to reduce computational burden as it can be efficiently solved online. The controller performances are validated on hydrographic process for which the levels of their two bottom tanks are accurately tracked to prescribed references trajectories.

Keywords: Hybrid system, Quadratic programming, NGPC, Constraints, Hydrographic process.

1. INTRODUCTION

Systems involving continuous and discrete dynamics are called hybrid systems. Their applications have arisen in manufacturing systems, automobile control, chemical and hydraulic process industry, and electrical circuits…

In recent years, a great effort has been devoted to developing optimal control theory for hybrid systems. A general optimal control framework for such systems was established in [1]. Furthermore, one can distinguish three major approaches to resolve optimal control problem: the gradient technique [2-6], the dynamic programming theory [7] and Pontruyagin maximum principle [8-11]. In [7], the authors develop a search algorithm based on dynamic programming to obtain derivatives of the cost function with respect to the switching instants. Then the optimal solution is obtained by a nonlinear search method. In [2], an optimal control of switched systems based on parameterization of the switching instants is studied after decomposing the problem in two stages. In the first one, a conventional optimal control problem is used in order to find the optimal cost for a given sequence of active subsystems and switching instants. In the second stage, a nonlinear optimization problem leads to the local optimal switching instants as given in [3, 4]. Moreover, variants of the maximum principle were derived in [10, 12, 13].

However, solving such problems is generally very difficult and the optimal control for general hybrid system still far from trivial since one has to deal not only with the infinite dimensional optimization problem related to the continuous dynamics, but also with a potential combinatory explosion related to the discrete part [14]. Furthermore, the optimization problem often becomes non convex and more complex especially in the presence of constraints and nonlinear systems characteristics, which makes the computational burden of those techniques grows exponentially with the decision variable. So, various restrictions have been imposed in order to obtain computationally feasible results.

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To overcome these limitations, many researches were oriented to the Model Predictive Control (MPC) or Receding Horizon Control (RHC), which is becoming a successful standard control technique in the process industries. This success is mainly due to its ability to deal with constraints on inputs and outputs. In addition to its great success for linear system [15], some contributions of synthesis NGPC law to different classes of nonlinear systems in continuous time have been achieved [16, 17]. In extension of these alternatives, Chen in [18, 19] gives a closed form solution for optimal NGPC. An analytic solution of predictive controller is given explicitly. Consequently, the online optimization is not required and the stability of the closed loop is guaranteed.

In this paper, within the framework of predictive control proposed by Chen, a closed loop predictive control strategy for switched nonlinear system is presented. The problem is then to find a switching strategy that minimizes a non convex cost function in *Bolza* form. As an application, a hydrographic process with four interconnected tanks is studied.

The structure of the paper is as follows. In section II, we formulate the optimal control problem. In section III, we introduce analytic NGPC law developed through the minimization of a receeding horizon performance criterion. In section IV, a brief description of the quadruple tank process is presented and the simulation results are illustrated. Finally, some conclusions are given.

2. PROBLEM FORMULATION

2.1 Nonlinear Switched Hybrid Systems

In our study, we consider particular system class, input affine nonlinear continuous switched hybrid systems described by the following subsystem:

$$\begin{cases} \dot{x}(t) = f_q(x(t)) + g_q(x(t))u_q(t) \ \text{ if } t \in \left[\tau_i, \tau_{i+1}\right[, \quad i = 1, ..., N, \\ y_q(t) = \left[y_1, ..., y_l\right]^T = h_q(x(t)) \end{cases} \tag{1}$$

where $f_q : \mathbb{R}^n \to \mathbb{R}^n, g_q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ $,h_q : \mathbb{R}^n \to \mathbb{R}^l$.

The following assumptions will be made.

A1: f_q and g_q are continuous functions, and have a continuous partial derivative with respect to x.

A2: $y_q(t)$ is sufficiently many times continuously differentiable with respect to time t.

A3: Each of the system output $y_q(t)$ has the same well defined relative degree r [18].

The control signal $u_q(t) \in U_q$ is a set of continuous function for $t \in \left[t_0, t_f\right]$ that take values in $\mathbb{R}^m$.

The function $q(t)$ is a piecewise constant function defined as:

$$q(t) = \begin{cases} q_0 & \text{if } t \in [\tau_0, \tau_1[\\ q_1 & \text{if } t \in [\tau_1, \tau_2[\\ \vdots & \quad \vdots \\ q_k & \text{if } t \in [\tau_k, \tau_{k+1}[\\ \vdots & \quad \vdots \end{cases} \tag{2}$$

Note that $q(t) \in \Pi$; with $\Pi = \{1, 2, \ldots, N\}$ presents a set of all admissible switching strategies. A switching sequence in $t \in [t_0, t_f]$ regulates the sequences of active subsystems and is defined as $\sigma = \{(\tau_k, q_k)\}_{k=1}^{N}$ and let τ_k, be a monotone nondecreasing finite sequence of time-points such that $t_0 = \tau_0 \le \tau_1 \le \ldots \le \tau_k \le t_f$.

We note that (τ_k, q_k) indicates at instant τ_k the system switches from subsystem q_{k-1} to subsystem q_k and during the time interval $[\tau_k, \tau_{k+1}[$, the subsystem q_k is active.

We exclude the zeno phenomena in the interval $[t_0, t_f]$.

2.2 Optimal Control Problem

We consider the switched nonlinear dynamic hybrid system described by (1).

Given a fixed time interval $[t_0, t_f]$ and a modal sequence $F_q = \{f_q + g_q u_q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n, q \in \Pi\}$, we need to find both continuous inputs $u_q(t) \in U_q$ and a switching strategy $q(t) \in \Pi$ such that the following cost functional:

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L_q(x, u_q) dt \tag{3}$$

is minimized.

With $L_q(x, u_q) = \frac{1}{2}\left(\left(y_q - y_{ref}\right)^T Q \left(y_q - y_{ref}\right) + u_q^T R u_q\right)$;

Where y_q and y_{ref} are the output vectors and the prescribed reference trajectory respectively. Q and R are positive semi-defined and positive defined matrices respectively. $\psi(x(t_f)) : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ is a terminal penalty.

3. CONSTRAINED NGPC FOR NONLINEAR SWITCHED HYBRED SYSTEMS

Applying the nonlinear predictive control to nonlinear switched hybrid systems leads to some results. So, the predictive control strategy aims to solve at each instant t the optimal control problem (4) yielding the optimal strategy of switching time $q(t) \in Q$ and an optimal continuous control $u_q(x(t), T_p(t)) \in U_q$ when $T_p(t) = N_p \times t_s$; N_p and t_s are the length of prediction and the sampling time respectively. The control problem is given as follow:

$$J\left(x(t),q(t)\right)=\psi\left(x(t+T_p(t))\right)+\int_t^{t+T_p(t)} L_q\left(x(\tau),u_q(\tau)\right)d\tau \tag{4}$$

subject to (1) and $q(t)\in\Pi$.

$u_q(x(t),T_p(t))$ is the optimal control profile defined on the interval $\left[t,t+T_p(t)\right]$ and only the first portion of the optimal profile is applied over $\left[t,t+t_s\right]$. At next instant $t+t_s$, a new optimal control problem is solved yielding the optimal solution $u_q(x(t+t_s),T_p(t+t_s))$ and its first portion is applied over $\left[t+t_s,t+2t_s\right]$. The procedure is repeated until the end of the horizon of time t_f. According to this, we are looking for the strategy of switching time and the continuous control which minimize (4). If we consider $T_p(t)$ the prediction horizon and τ_i the switching instant, so $\tau_i\in\left[T_p(t),T_p(t+t_s)\right]$. The solution of (4) is obtained by an exhaustive research which consists in comparing the cost for almost $Card(\Pi)$ trajectories (the complexity is linear in the number of the set Π) i.e the system switches from a model to another according to the one which gives the greater cost and the correspondent model is activated over $\left[t,t+t_s\right]$ and so on.

In order to resolve the optimization problem, the controller is designed such that the closed-loop system is asymptotically stable and the output, $y_q(t)$, of the nonlinear switched hybrid system (1) optimally tracks a prescribed reference trajectory, $y_{ref}(t)$. The control signal is defined as the solution of receeding horizon index performance minimization at every sampling time instant t_s.

$$J(x(t),q)=\frac{1}{2}\int_t^{t+T_p(t)}\left(\left(y_q(t+\tau)-y_{ref}(t+\tau)\right)^T Q\left(y_q(t+\tau)-y_{ref}(t+\tau)\right)\right)d\tau \tag{5}$$

We assume that both the output $y_q(t)$ and the reference trajectory $y_{ref}(t)$ verify assumption ***A2.***

To avoid the difficulties in solving PDEs in classic optimal control theory, the predictive values of the output $y_q(t+\tau)$ at time τ is given by Taylor series expansion [18].

$$y_q(t+\tau)=\Gamma(\tau)Y_q(t) \tag{6.a}$$

In the same

$$y_{ref}(t+\tau)=\Gamma(\tau)Y_{ref}(t) \tag{6.b}$$

where

$$\Gamma(\tau)=\left[1\ \tau\ \ldots\frac{\tau^{\mathrm{r}}}{\mathrm{r}!}\right] \tag{6.c}$$

$$Y_q(t)=\left[y_q(t)\ \dot{y}_q(t)\ \ldots y_q^{[r]}(t)\right]^T \tag{6.d}$$

$$Y_{ref}(t)=\left[y_{ref}(t)\ \dot{y}_{ref}(t)\ \ldots y_{ref}^{[r]}(t)\right]^T \tag{6.e}$$

Repeated differentiation up to r times of the output $y_q(t)$ with respect to time, together with repeated substitution of system (1) gives:

$$\begin{aligned} &\dot{y}_q(t) = L_{f_q} h_q(x), \\ &\vdots \\ &y_q^{[r-1]}(t) = L_{f_q}^{r-1} h_q(x), \\ &y_q^{[r]}(t) = L_{f_q}^{r} h_q(x) + L_{g_q} L_{f_q}^{r-1} h_q(x) u_q(t), \end{aligned} \tag{7.a}$$

where

$$h_q(x) = \left[h_{1,q}(x), \ldots, h_{l,q}(x)\right]^T \tag{7.b}$$

The standard Lie notation is used in this paper and the control order ρ is chosen to be zero.

According to (5) we can write:

$$J = \frac{1}{2}\left(Y_q(t) - Y_{ref}(t)\right)^T \overline{\Gamma}(T_p(t))\left(Y_q(t) - Y_{ref}(t)\right) \tag{8.a}$$

where

$$\overline{\Gamma}(T_p(t)) = \int_t^{t+T_p(t)} \Gamma^T(\tau)\Gamma(\tau) d\tau \tag{8.b}$$

The ij^{th} element of the square matrix $\overline{\Gamma}$ is in the form

$$\Gamma_{(i,j)} = \frac{\overline{T}_p^{i+j-1}}{(i-1)!(j-1)!(i+j-1)} \quad , \qquad i, j = 1, \ldots, r + \rho + 1 \tag{9}$$

In other hand, let

$$Y_q(t) - Y_{ref}(t) = M_{r,q} + \begin{bmatrix} 0_{r\times 1} \\ N_q(\overline{u}_q) \end{bmatrix} \tag{10.a}$$

where $M_{r,q} \in \mathbb{R}^{lr}$ is given by:

$$M_{r,q} = \begin{pmatrix} h(x) - y_{ref}(t) \\ L_{f_q}^1 h(x) - y_{ref}^{[1]}(t) \\ \cdots \\ L_{f_q}^{r-1} h(x) - y_{ref}^{[r-1]}(t) \end{pmatrix} \tag{10.b}$$

and

$$N_q(\overline{u}_q) = L_{g_q} L_{f_q}^{r-1} h_q(x) \overline{u}_q \tag{10.c}$$

After simple computation, the receeding horizon index performance can be rewriting as:

$$J = \frac{1}{2}\left(M^T\overline{\Gamma}M + 2M^T\Gamma_{r+1}N(\overline{u}) + N^T(\overline{u})\Gamma_{r+1,r+1}N(\overline{u})\right) \tag{11}$$

where Γ_{r+1} is the $(r+1)^{th}$ column of matrix $\overline{\Gamma}$ and $\Gamma_{r+1,r+1}$ its $(r+1,r+1)^{th}$ element.

To avoid acute peaks in control signal, we should to saturate manipulated variable. Thus, in presence of constraints, the optimum predictive control trajectory is henceforth defined through the on-line solution of quadratic programming problem. This convex optimization problem can be efficiently solved through a trivial on-line computation demand.

Thus, we can reformulate the predictive control law as a solution of the following quadratic programming optimization problem

$$\min_{\overline{u}} \ \frac{1}{2}\overline{u}^T\left(\left(L_{g_q}L_{f_q}^{r-1}h_q(x)\right)^T\Gamma_{r+1,r+1}\left(L_{g_q}L_{f_q}^{r-1}h_q(x)\right)\right)\overline{u} + M_q^T\Gamma_{r+1}\left(L_{g_q}L_{f_q}^{r-1}h_q(x)\right)\overline{u} \tag{12}$$

Subject to $A\overline{u} \le B$

where $A = [I_{l\times l}, -I_{l\times l}]^T$, $B = [U_{\max}, -U_{\min}]^T$, $U_{\max} = [u_{1,\max}, \cdots, u_{l,\max}]^T$ and $U_{\min} = [u_{1,\min}, \cdots, u_{l,\min}]^T$.

Using *quadprog* function integrated in *Matlab* environment, we can easily compute online the optimal control signal.

4. APPLICATION

4.1 Description of the Quadruple Tank Process

The four tanks system is a part of research unit on Numeric Control of the Industrial Processes (CONPRI) which is located in Monastir Engineering School [20]. The process to be controlled is shown in fig. 1 and consists of two pumps used to convey liquid from basin into two upper overhead tanks. Liquid flows from the two upper tanks to the two bottom ones through four pipes which have on/off valves in order to activate or to block the lines.

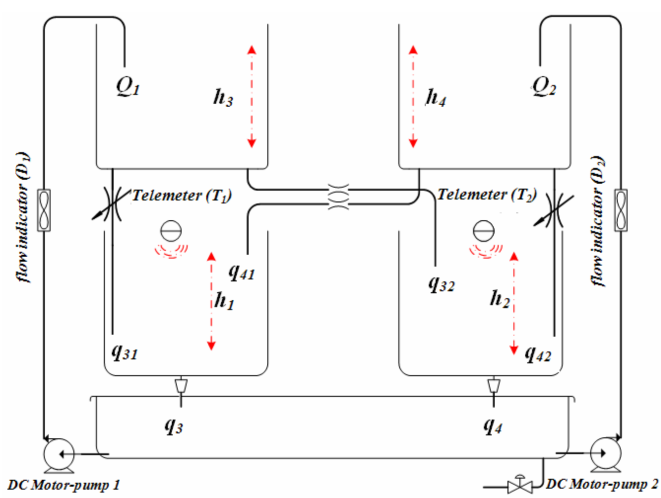

Fig. 1. Schematic of the four tanks system

Considering some possible combinations of the valves states $(0/1)$, three models are generated. The first one is given when all the valves are activated; the second and the third ones are obtained by setting respectively *off* and *on* only the interconnected valves. So, using mass balance and Bernoulli's law [20], the process is modelled by:

$$sys1:\begin{cases}\dot{z}_1 = -c_1\sqrt{z_1} + c_2\sqrt{z_3} + c_3\sqrt{z_4}\\ \dot{z}_2 = -c_4\sqrt{z_2} + c_5\sqrt{z_3} + c_6\sqrt{z_4}\\ \dot{z}_3 = -c_7\sqrt{z_3} + c_8u_1\\ \dot{z}_4 = -c_9\sqrt{z_4} + c_{10}u_2\\ y(t) = [z_1, z_2]^T = h_1(z(t))\end{cases} \quad (13.a)$$

$$sys2:\begin{cases}\dot{z}_1 = -c_1\sqrt{z_1} + c_2\sqrt{z_3}\\ \dot{z}_2 = -c_4\sqrt{z_2} + c_6\sqrt{z_4}\\ \dot{z}_3 = -c_2\sqrt{z_3} + c_8u_1\\ \dot{z}_4 = -c_6\sqrt{z_4} + c_{10}u_2\\ y(t) = [z_1, z_2]^T = h_2(z(t))\end{cases} \quad (13.b)$$

$$sys3:\begin{cases}\dot{z}_1 = -c_1\sqrt{z_1} + c_3\sqrt{z_4}\\ \dot{z}_2 = -c_4\sqrt{z_2} + c_5\sqrt{z_3}\\ \dot{z}_3 = -c_5\sqrt{z_3} + c_8u_1\\ \dot{z}_4 = -c_3\sqrt{z_4} + c_{10}u_2\\ y(t) = [z_1, z_2]^T = h_3(z(t))\end{cases} \quad (13.c)$$

where z_i denotes the liquid level of tank i, $i \in \{1,..,4\}$, u_j the speed setting of pump j, and c_i, $i = 1..10$ are the system parameters values given in table 1.

We assume that the relation between the output flow of the pumps and the correspondent input signal is considered linear.

TABLE 1: Process's parameters Values

Parameters	Values
c_1, c_4	0.01462
c_2, c_6	0.01462
c_3, c_5	0.0532
c_7, c_9	0.1993
c_8, c_{10}	0.1830

4.2 Simulation results

The goal is to control the levels of the two bottom tanks in order to track a prescribed profile. According to (7.a), the relative degree of the system is $r = 2$ and the control order is chosen to be $\rho = 0$. The predictive time is chosen to be $T_p = 21\,\mathrm{s}$, the sample time is $t_s = 1\,\mathrm{s}$, $t_f = 1000\,\mathrm{s}$ and the initials conditions are $z(0) = [0.5\ 0.5\ 2.5\ 2.5]^T \times 10^{-2} m$, $Q = 0.1 \times I_{2\times2}$ and $R = 0.1 \times I_{2\times2}$. Inequality constraints are $A = \left[I_{2\times2}; -I_{2\times2}\right]$ and $B = u_{\max} \times [1\ 1\ 0\ 0]^T$, where $u_{\max} = 15V$ and $I_{2\times2}$ is the identity matrix.

The simulations have been carried out in Matlab environment to verify the performance of the proposed controller.

Fig. 2 and Fig. 3 give the responses (levels of two bottom tanks) of the NGPC for the quadruple tank system with references. As shown in these graphs, the tracking performance of the proposed NGPC system is satisfactorily achieved which is proved by the evolution of performance criterion in Fig. 4. The switching profile is given by Fig. 5. The resulting controls time through *QP* algorithm are given in Fig. 6. These signals stay inside limits of imposed input constraints.

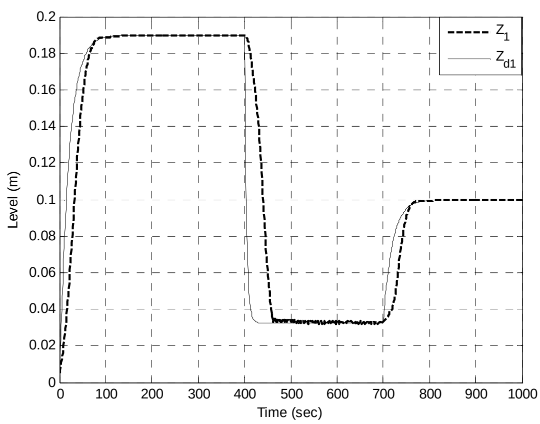

Fig. 2. Evolution time of the bottom left tank's level

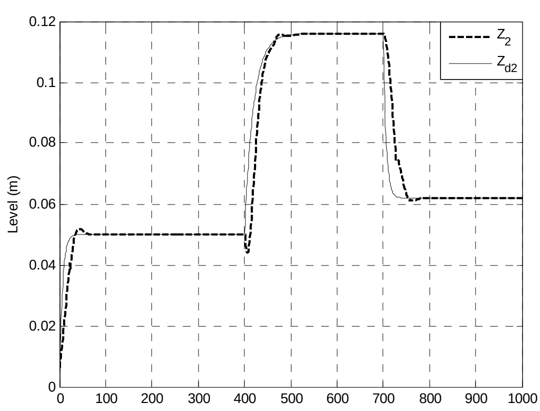

Fig. 3. Evolution time of the bottom right tank's level

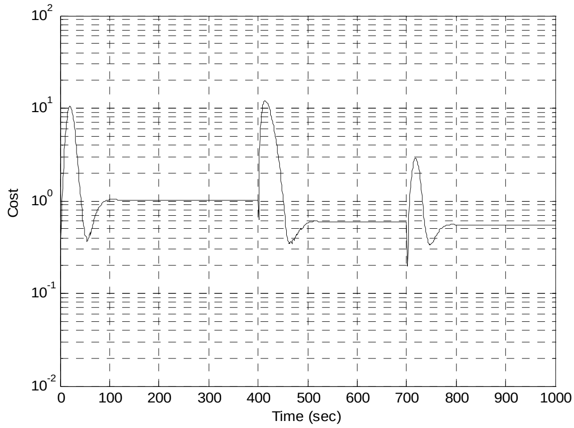

Fig. 4. Evolution time of the performance criterion

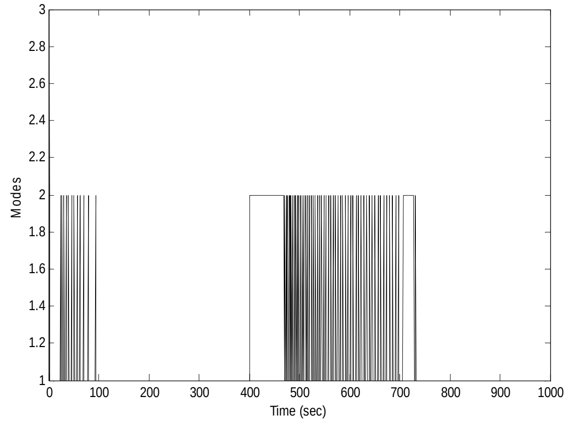

Fig. 5. Switching profile

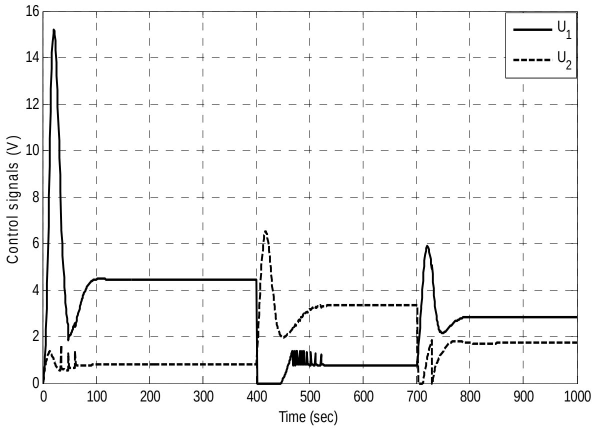

Fig. 6. Optimal control signals

4.3 Remarks

It appears that the performance tracking of the desired trajectory is sensitive to the predictive horizon choice. In fact, large value of T_p leads to the stability, but it grows the settling time. So, this parameter should be chosen carefully.

The main feature of predictive control strategy is its possibility to take implicitly into account input constraints. This can replace control saturation, which generally is given for other control techniques. Furthermore, handling constraints implicitly in control law, bring to the actuators more flexibility in the decision variable of control.

5. CONCLUSION

In this paper, a predictive control strategy for nonlinear switched hybrid systems is presented. A nonlinear predictive control law is synthesized for each subsystem as a solution of QP optimization problem. It arises from the receeding horizon index minimization which takes into account implicitly the control constraints. The switching strategy obtained by an exhaustive search permits to improve the minimisation of the cost performance criterion. Indeed, optimal switching instants are obtained without using complex computation by differentiation of cost function. An application to MIMO nonlinear system, such as the hydrographic process is outlined. It's shown that their two bottom levels liquid are efficiently controlled to track prescribed references trajectories.

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