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Nonlinear Observers-Based State Feedback Control: Application to an Inverted Pendulum

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In This paper, we present a robust high gain output feedback control for a class of uniformly observable and controllable nonlinear systems. Our main goal here is to apply an output feedback controller which regulates the output of the plant to a constant reference. While the smooth state feedback control law is constructed based on the use of a high gain sliding mode controller, the estimation of the non measured states is carried out by a sliding mode observer which is compared to a high gain observer. It's shown that the resulting closed loop is asymptotically stable, and the performance recovery includes asymptotic stability of the origin, the region of attraction and trajectories. Furthermore, the system equation is augmented through a filtered integral action which is incorporated into the control law to ensure asymptotic rejection of step like states and/or output disturbances, and measurement noise. As a particular application, we consider the open loop unstable inverted pendulum on a cart. Finally, computer simulations are developed for showing the robustness of the controller including sliding mode observer against step like disturbance, noisy measurements and parametric uncertainties.

Keywords: Output Feedback Control, Nonlinear Observers, High Gain, Sliding Mode, Robustness, Inverted Pendulum.

1. INTRODUCTION

During the last four decades, state feedback notion has attracted the attention of numerous researchers and has been very widly used in many control strategies. Due to the easiness of the control principle and its implementation, several methods were proposed to put some control strategies under a state feedback, as for example, predictive control [1-3], adaptive control [4, 5], sliding mode control [6, 7] and high gain control [8-11].

As high gain controller is computationally efficient, many works were proposed to put such control strategy under state feedback form. Authors in [10] investigated stabilization of single input nonlinear systems by switching between two separate nonlinear feedback controllers. While the first is pseudo-linear, the second is a high gain controller. Freidovich and Khalil [11] considered high gain output feedback stabilization of uniformly observable uncertain nonlinear systems when the uncertain parameters belong to a known but comparably compact set. Unfortunately, to calculate and implement such process control strategies, the information about the internal state of the process is necessary. Consequently, the presence of unknown state variables becomes a difficulty which can be overcome with the inclusion of an appropriate state estimator.

Estimation theory for linear systems, in the deterministic case, is well developed and has proved its effectiveness since the synthesis of Luenberger observer [12] and the Kalman filter [13] in a stochastic environment. Therefore, the development of suitable algorithms to perform the estimation for nonlinear systems has captured the attention of many researchers. In this sense, several techniques have been introduced in the literature such as,

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for example, Luenberger like observer [3], extended Kalman filter [14, 15], receding horizon observer [16], sliding mode observer [17, 18] and high gain observer [19-21].

From the most interesting items in the case of output feedback, the stabilization of the closed loop system. While for controllable and detectable linear systems, the separation principle implies that the stability of the closed loop follows from the individual stability of the controller and the observer [22], this problem is still not evident for nonlinear systems and is an open research axe in the case of nonlinear systems despite the existence of many works accomplished in this field. Bounit and Hammouri [23] studied the problem of global feedback stabilization via bounded smooth state feedback of infinite dimensional linear systems. In [24], Atassi and Khalil developed separation principle for the stabilization of a class of nonlinear systems having a chain or more of integrators in their structure in a generic form that can be applied to any globally bounded stabilizing state feedback control. In particular, they presented a separation theorem that is independent of the state feedback control and is derived under the least restrictive assumptions. Moreover, they demonstrated that the performance recovery achieved with sufficiently fast high gain observer is more than just asymptotic stability recovery. It includes recovery of region of attraction and trajectories achieved under state feedback. Later, Atassi and Khalil [25] are interested in feedback controllers that achieve boundedness of trajectories but not necessarily with convergence to an equilibrium point. After that, Khalil [26] and Mahmoud and Khalil [27], studied the robustness of state feedback control based high gain observer to unmodeled actuator and sensor dynamics.

Recently, Freidovich and Khalil proved, in [11], that for a certain class of nonlinear systems, in which uncertainty is due to unknown parameters, it has been possible to combine nonlinear robust or adaptive control techniques with high gain observer to derive output feedback controller that ensure stabilization or practical stabilization for given compact sets of initial conditions. Furthermore, to improve robustness against step like disturbances, many control schemes incorporated an integral action. So, Yang and Lin [28] elaborated a smooth state feedback control law based on the tool of adding a power integrator. Authors, in [29], presented a sliding mode control with an integral action. In reference [30], reliable stabilization with integral action was studied in the case of linear MIMO two channel decentralized control system. This technique is used later in [31] with the same class systems to achieve closed loop stability and robust asymptotic tracking of step input references with zero steady state error. In [32], Seshagiri and Khalil designed an output feedback continuous sliding mode control with integral action for MIMO minimum phase nonlinear systems with a well defined relative degree, to improve transient performance. In [33], Mahmoud and Khalil used integral control to ensure asymptotic regulation, in the case of constant references, for a SISO minimum phase nonlinear system that is transformable into the normal form uniformly in a set of constant disturbances and uncertain parameters. The introduction of the integrator creates an equilibrium point at which the tracking error is zero for all possible parameters and/or disturbances that belong to a known compact set.

In this paper, we will be based on the works reported in [8] and [34] where authors proposed a high gain output feedback control for two classes of nonlinear uniformly observable and controllable systems with an admissible tracking capability. In the proposed control law, a filtered integral action is incorporated to carry out asymptotic rejection of step like disturbances and measurement noise. According to our case, we will adopt the nonlinear systems class considered in [34] and we will apply the resulting state feedback controller based on an improved estimation strategy. In fact, we propose to replace the HGO used in [8] and [34] by a sliding mode observer. We show that, with such choice, we perform well the control signal input and we minimize the estimation error. Also, we ensure that the separation theorem as presented by Atassi and Khalil [24] is still held for our case and we guarantee, then, the performance recovery of the closed loop system. To illustrate the advantage of our estimator choice, we applied the resulting control law to stabilize an open loop unstable nonlinear process represented by an inverted pendulum on a cart, around an upright position.

The rest of this paper is organized as follows. The problem formulation is presented in the next section when we introduce the nonlinear system class considered. Output feedback control based high gain observer is treated in section 3. Some particular design functions are introduced in section 4, while the incorporation of a filtered integral action is reported in section 5. The application of the control law based high gain and sliding mode observer to regulate an inverted pendulum on a cart around an upright position is done in section 6. Finally, conclusions are presented in the last section.

2. PROBLEM FORMULATION

We seek an admissible regulation problem for a class of uniformly observable and controllable nonlinear systems. Therefore, we will rely on the works of Farza et al. [8] and M'Saad [34] who synthesized a high gain output feedback control law incorporating a filtered integral action for two classes of nonlinear systems. We propose, in what follows, a brief description of the principle steps to obtain the resulting control law for the class of nonlinear systems considered in [34] which can be described by the following state and output equations:

$$\begin{cases} \dot{x} = Ax + Bb(x)u + \varphi(x), \\ y = Cx = x^1 \end{cases} \tag{1}$$

with:

$$A = \begin{pmatrix} 0 & I_{n-p} \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0_p \\ 0_p \\ \vdots \\ I_p \end{pmatrix}, C = \begin{pmatrix} I_p & 0_p & \dots & 0_p \end{pmatrix},\ x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^q \end{pmatrix}, \varphi(x) = \begin{pmatrix} \varphi^1(x^1) \\ \varphi^2(x^1, x^2) \\ \vdots \\ \varphi^{q-1}(x^1, \dots, x^{q-1}) \\ \varphi^q(x) \end{pmatrix},$$

where the state $x \in \vartheta$ a compact set of $\Re^n$ with $x^k \in \Re^p$, $k = 1, \dots, q$, the input $u \in U$ a compact set of $\Re^m$ with $m \geq p$ and $b(x)$ is $p \times m$ rectangular matrix.

The control problem considered therein consists in the convergence of the output $x^1(t)$ of the system to a desired output trajectory that will be denoted by $\left\{x_d^1(t)\right\} \in \Re^p$, i.e.

$$\lim_{t \to \infty} \left(x^1(t) - x_d^1(t)\right) = 0 \tag{2}$$

To deal with this problem, we assume the following assumptions:

A1: The matrix $b(x)$ is lipschitz in x on ϑ and is of full rank. Moreover, it exists two positive scalars α and β such that $\forall x \in \vartheta$, we have:

$\alpha^2 I_p \leq b(x)(b(x))^T \leq \beta^2 I_p$,

A2: The function φ is lipschitz in x on ϑ.

Besides, we assume that the derivatives of the desired output sequence up to order n are available. To the sequence $\{x_d^1(t)\} \in \Re^p$, correspond an input sequence and a state trajectory of the system that will be denoted, respectively, by $\{u_d(t)\} \in \Re^m$ and $\{x_d(t)\} \in \Re^n$. A resulting model reference for the class of nonlinear systems considered therein can be done by:

$$\begin{cases} \dot{x}_d = Ax_d + Bb(x_d)u_d + \varphi(x_d), \\ y_d = Cx_d \end{cases} \tag{3}$$

The model reference of the state $x_d \in \Re^n$ and the associated input $u_d \in \Re^m$ can be determined recursively from the model reference (3) as follows:

$$\begin{cases} x_d^1 = y_d \\ x_d^k = \dot{x}_d^{k-1} - \varphi^{k-1}\left(x_d^1 \quad \dots \quad x_d^{k-1}\right) \quad for \quad k \in [2,q] \\ u_d = \left(b(x_d)\right)^+ \left(\dot{x}_d^q - \varphi(x_d)\right) \end{cases}$$

Moreover, when the reference trajectory is smooth enough, we show that the state model reference and the input can be determined recursively from the reference trajectory and its successive derivatives, as follows:

$$x_d^k = g^k\left(y_d, y_d^{(1)}, \dots, y_d^{(k-1)}\right) \quad for\ k \in [1,q] \tag{4}$$

with $y_d^{(i)} = \dfrac{d^i y_d}{dt^i}$,

where the functions g^k are given by:

$$\begin{cases} g^1(y_d) = y_d, \\ g^k\left(y_d, y_d^{(1)}, \dots, y_d^{(k-1)}\right) = \sum_{j=0}^{k-2} \dfrac{\partial g^{k-1}}{\partial y_d^j}\left(y_d, \dots, y_d^{(k-2)}\right) y_d^{(j+1)} - \varphi^{k-1}\left(g^1(y_d), \dots, g^{k-1}(y_d, \dots, y_d^{(k-2)})\right) \\ for\ k \in [2,q] \end{cases}$$

The original tracking problem of the output given by relation (2) can be extended to include tracking of the state trajectory of the system given by:

$$\lim_{t\to\infty}(e(t)) = \lim_{t\to\infty}\left(x(t) - x_d(t)\right) = 0 \tag{5}$$

This problem can be viewed as a regulation problem of the error system between the original system (1) and the model reference (3):

$$\begin{cases} \dot{e} = Ae + B(b(x)u(x) - b(x_d)u(x_d)) + \varphi(x) - \varphi(x_d) \\ e_m = y - y_d \end{cases} \tag{6}$$

3. OUTPUT FEEDBACK CONTROL

Output feedback control is a state feedback control where the only measurable state introduced in the control law is the output of the system and other states are estimated on line. The synthesis of such controller is conforming to the separation theorem. The estimation of non available states is accomplished using two approaches. While the first one is based on the high gain principle as presented by Farza et al. [19], the second strategy which we adopted uses the sliding mode strategy [17]. The resulting state feedback control for the nonlinear systems class considered in this work is then given by:

$$u(\hat{x}) = \left(b\left(\hat{x}\right)\right)^{+}\left(\dot{x}_d^q - \varphi^q(x_d) + \nu(\hat{e})\right) \tag{7}$$

with:

$$\nu(\hat{e}) = -\lambda B^T \Delta_\lambda^{-1} K_c\left(\overline{S}\Delta_\lambda\left(\hat{x} - x_d\right)\right) \tag{8}$$

where $\left(b\left(\hat{x}\right)\right)^{+}$ denotes the right inverse of $b\left(\hat{x}\right)$ that we can obtain if assumption A1 holds, λ is a positive scalar, Δ_λ is a block diagonal matrix given by:

$$\Delta_\lambda = diag\left(I_p, \frac{1}{\lambda}I_p, \ldots, \frac{1}{\lambda^{q-1}}I_p\right), \tag{9}$$

$\overline{S}$ is the unique solution of the following algebraic Lyapunov equation:

$$\overline{S} + A^T\overline{S} + \overline{S}A = \overline{S}BB^T\overline{S} \tag{10}$$

and $K_c : \Re^n \rightarrow \Re^n$ is a bounded design function satisfying the following property:

$$\forall \xi \in \Omega \quad we\ have \quad \xi^T BB^T K_c(\xi) \geq \frac{1}{2}\xi^T BB^T \xi \tag{11}$$

where Ω is a compact subset of $\Re^n$.

Remark 1:

Since the matrix S, the solution of the following algebraic Lyapunov equation:

$$S + A^T S + SA - C^T C = 0 \tag{12}$$

is unique and positive definite as proved by Gauthier et al. [12], and taking into account the structure of the matrix B and C, one can conclude that $\overline{S}$ is also unique and positive definite, and it's given by:

$$\overline{S} = TS^{-1}T \tag{13}$$

where

$$T = \begin{pmatrix} 0_p & \ldots & 0_p & I_p \\ \vdots & 0_p & I_p & 0_p \\ 0_p & I_p & 0_p & \vdots \\ I_p & 0_p & \ldots & 0_p \end{pmatrix}$$

Using some useful algebraic manipulations as in Farza et al. [21] yields:

$$B^T \bar{S} = CS^{-1}T = \left[C_q^q I_p \quad C_q^{q-1} I_p \quad \ldots \quad C_q^1 I_p \right]$$

Taking into account the structure of the output feedback control law given by (7), a high gain observer elaborated by Farza et al. [19] for the system (1) is done by the following relation:

$$\dot{\hat{x}} = A\hat{x} + B\nu(\hat{e}) + \varphi(\hat{x}) - \varphi(x_d) - \theta\Delta_\theta S^{-1}C^T C\varepsilon \tag{14}$$

where $\hat{e} \in \Re^n$ is the estimation of the state tracking error e, Δ_θ is a diagonal matrix definite as the matrix Δ_λ for the real $\theta > 0$, the matrix S is the solution of the equation (12), and $\varepsilon = (\hat{x} - x)$ denotes the state estimation error.

Taking into account the error system (6), (7), (8) and (14) given the output feedback control scheme, the resulting control law can be described by the equations of the tracking error estimation and the equations of the state estimation error given, respectively, by:

$$\begin{aligned} \dot{\hat{e}} &= A\hat{e} + B\nu(\hat{e}) + \varphi(\hat{e} + x_d) - \varphi(x_d) - \theta\Delta_\theta S^{-1}C^T C\varepsilon \\ \dot{\varepsilon} &= A\varepsilon + B\left(b(\hat{x}) - b(x)\right)u(\hat{x}) + \varphi(\hat{x}) - \varphi(x_d) - \theta\Delta_\theta S^{-1}C^T C\varepsilon \end{aligned} \tag{15}$$

According to theorem 4.1 of [34], one can conclude that state feedback control based high gain observer as mentioned above is globally stable and ensure asymptotic and exponential convergence of the tracking error provided that assumptions A1 and A2 hold. Besides, the high gain principle allows recovering the separation theorem for the nonlinear system class considered therein. For demonstration and more details, readers can go back to references [8] and [34].

4. PARTICULAR DESIGN FUNCTIONS

In the resulting control law given by relations (7) and (8), we show that the controller involves a bounded design function which provides a unified framework for the high gain control design. Some useful design functions satisfying inequality (11) are given below:

***** High gain design function given by:

$$K_c(\xi) = k\xi \tag{16}$$

where $k \geq \dfrac{1}{2}$ is a positive scalar.

This function satisfies property (11) on $\Re^n$.

The sliding mode design function is given by:

$$K_c(\xi) = k\, sign(\xi) \tag{17}$$

where k is a positive scalar and sign is the usual signum function given by:

$$\operatorname{sgn}(\xi) = \begin{cases} -1 & if \quad \xi < 0 \\ 0 & if \quad \xi = 0 \\ 1 & if \quad \xi > 0 \end{cases}$$

In the case of bounded input bounded state systems, condition (11) holds for high values of k.

The design function that is commonly used in the smooth sliding mode practice, namely:

$$K_c(\xi) = k\ \tanh(a\xi) \tag{18}$$

where tanh denotes the hyperbolic tangent function, and k and a are positive scalars. Property (11) holds for high value of parameter a. In fact:

$$\lim_{a\to\infty}(\tanh(a\xi)) = sign(\xi) \tag{19}$$

Such function is employed to overcome the chattering phenomena usually induced by the use of the sign function.

5. INCORPORATION OF A FILTERED INTEGRAL ACTION

To improve the robustness of the closed loop system against step like disturbances and noise measurement, authors in [8] incorporated, in the control law, a filtered integral action. So, the system equation is augmented through the introduction of suitable state variables as follows:

$$\begin{cases} \dot{\sigma}^f = e^f \\ \dot{e}^f = -\Gamma e^f + \Gamma e^1 \end{cases} \tag{20}$$

where $\Gamma = diag\{\gamma_i\}$ is a design parameter associated to the integral action. The resulting augmented model is then given by:

$$\begin{cases} \dot{e}_a = A_a e_a + B_a\left(b(e + x_d)u(e_a + x_{da}) - b(x_{da})u_{da}\right) + \psi(e_a + x_{da}) - \psi(x_{da}), \\ y_a = \sigma^f \end{cases} \tag{21}$$

where:

$$e_a = \begin{pmatrix} \sigma^f \\ e^f \\ e \end{pmatrix}, x_{da} = \begin{pmatrix} \sigma_d^f \\ y_d^f \\ x_d \end{pmatrix}, A_a = \begin{pmatrix} 0 & I_p & 0 \\ 0 & 0 & \Gamma \\ 0 & 0 & A \end{pmatrix}, B_a = \begin{pmatrix} 0_p \\ 0_p \\ B \end{pmatrix}, \psi(e_a) = \begin{pmatrix} 0_p \\ -\Gamma e^f \\ \varphi(e) \end{pmatrix}$$

From the fact that the synthesis model structure (21) is similar to the error system (6), the synthesis of the controller is the same. Then, a high gain output feedback control with a filtered integral action is given by:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu(\hat{e}_a) + \varphi(\hat{x}) - \varphi(x_d) - \theta\Delta_\theta S^{-1}C^T C\varepsilon \\ u(\hat{e}_a) = \left(b(\hat{e}_a + x_{da})\right)^+ \left(\dot{x}_d^q - \varphi^q(x_d) + \nu(\hat{e}_a)\right) \\ \nu(\hat{e}_a) = -\lambda\Gamma^{-1}B_a^T\Delta_{a\lambda}^{-1}K_{ac}\left(\overline{S}_a\Delta_{a\lambda}\Lambda\hat{e}_a\right) \end{cases} \tag{22}$$

with:

$$\hat{e}_a = \begin{pmatrix} \sigma^f \\ e^f \\ \hat{e} \end{pmatrix}, \Delta_{a\lambda} = diag\left(I_p, \frac{1}{\lambda} I_p, \ldots, \frac{1}{\lambda^q} I_p, \frac{1}{\lambda^{q+1}} I_p\right),\ \Lambda = diag(I_p, I_p, \Gamma, \ldots, \Gamma)$$

where $\overline{S}_a$ is the unique positive definite matrix solution of the following Lyapunov algebraic equation:

$$\overline{S}_a + A_a^T \overline{S}_a + \overline{S}_a A_a = \overline{S}_a \overline{B}_a \overline{B}_a^{\,T} \overline{S}_a \tag{23}$$

and $K_{ac} : \Re^{n+2p} \to \Re^{n+2p}$ is a bounded design function satisfying the following property:

$$\forall \xi_a \in \Omega \ \ we\ have \ \ \xi_a^T B_a B_a^T K_{ac}(\xi_a) \geq \frac{1}{2} \xi_a^T B_a B_a^T \xi_a \tag{24}$$

where Ω is a compact set of $\Re^{n+2p}$.

It can be shown that the resulting control law is globally stable and performs an asymptotic rejection of state and/or output step like disturbances [8].

6. APPLICATION

In this section, we propose to apply the sliding mode like output feedback control law presented above to regulate an (unstable) inverted pendulum on a cart around an upright position. The angle of the pendulum with the vertical axis is denoted by z_1. The input of the system is given by the force u which acts on the translation of the cart and is limited by ±10N.

The model equations of the cart-pendulum system is taken from [1] to which we added a step like disturbance $w(t)$. The resulting model is then given by the following set of state and output equations:

$$\begin{cases} \dot{z}_1 = z_2 + w(t) \\ \dot{z}_2 = \dfrac{f(z) + g(z)u}{h(z)}, \\ y = z_1, \end{cases} \tag{25}$$

with: $f(z) = ml\cos(z_1)\sin(z_1)z_2^2 - g(m+M)\sin(z_1),\ \ g(z) = \cos(z_1)\delta,$

$h(z) = ml\cos^2(z_1) - \frac{4}{3}(m+M)l,$

where z_1, z_2, m and l are, respectively, the position, the angular velocity, the mass and the length of the pendulum, and M is the mass of the cart. The angle z_1 is assumed to be the only measurable state.

One can, easily, verify that the model of the inverted pendulum (25) corresponds to the nonlinear system class given by relations (1) with:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix}, z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix}, \begin{cases} z^1 = z_1 \\ z^2 = z_2 \end{cases}$$

$$b(z) = \frac{g(z)}{h(z)}, \varphi(z) = \begin{pmatrix} \varphi^1(z^1) \\ \varphi^2(z^1, z^2) \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{f(z)}{h(z)} \end{pmatrix}$$

The control objective consists in the synthesis of an admissible control law to regulate the position of the pendulum at a desired constant value. According to system (3), the desired second state and the desired input can be computed, respectively, as follows:

$$\begin{aligned} z_d^2 &= \dot{z}_d^1 = y_d \\ u_d &= \left(g(z_d)\right)^{-1}\left(h(z_d)\dot{z}_d^2 - f(z_d)\right) \end{aligned} \tag{26}$$

For $z_d^1 = 0$, it's easy to show that $z_d^2 = 0$ and $u_d = 0$.

The state feedback control with a filtered integral action applied to the model of the inverted pendulum can be done by the following relations:

$$\begin{cases} u(z) = \left(b(z)\right)^{-1}\left(\dot{z}_d^2 - \varphi^2(z_d) + \nu(e_a)\right), \\ \nu(e_a) = -\lambda\Gamma^{-1}B_a^T\Delta_{a\lambda}^{-1}K_{ac}\left(\overline{S}_a\Delta_{a\lambda}\Lambda e_a\right) \end{cases} \tag{27}$$

where

$$e_a = \begin{pmatrix} \sigma^f \\ e^f \\ e \end{pmatrix}; \quad e = \begin{pmatrix} e^1 \\ e^2 \end{pmatrix}; \quad \begin{cases} e^1 = z^1 - z_d^1 = z^1 \\ e^2 = z^2 - z_d^2 = z^2 \end{cases},$$

$$B_a = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \quad \overline{S}_a = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 6 \\ 1 & 4 & 6 & 4 \end{pmatrix}; \quad \Delta_{a\lambda} = diag\left(1, \frac{1}{\lambda}, \frac{1}{\lambda^2}, \frac{1}{\lambda^3}\right), \ \Gamma = \gamma; \quad \Lambda = diag(1, 1, \gamma, \gamma);$$

The filtered integral action is governed by the following relations:

$$\begin{cases} \dot{\sigma}^f = e^f, \\ \dot{e}^f = -\gamma e^f + \gamma e^1, \end{cases} \tag{28}$$

and the design function satisfying property (25) is given by (18), i.e.:

$$K_{ac} : \Re^4 \to \Re^4, e_a \to K_{ac}(e_a) = k\tanh(ae_a)$$

A simple computation allows us to find the expression of $\nu(e_a)$ which is done by the following equation:

$$\nu(e) = -\frac{k\lambda^4}{\gamma}\tanh\left(a\left(\sigma^f + \frac{4}{\lambda}e^f + \frac{6\gamma}{\lambda^2}e^1 + \frac{4\gamma}{\lambda^3}e^2\right)\right) \tag{29}$$

However, the application of the state feedback control law given by equations (27) is not possible since such controller depends on the state z^2 which is not available for measurement. That's why one can refer to observation strategies to estimate the missing state. This step is accomplished by using two techniques. The first one is the high gain observer as proposed by Farza et al. [19]. Such observer for the inverted pendulum considered in this work is given by:

$$\begin{cases} \dot{\hat{z}}_1 = \hat{z}_2 - 2\theta\varepsilon^1 \\ \dot{\hat{z}}_2 = \dfrac{f(\hat{z}) + g(\hat{z})u(\hat{z})}{h(\hat{z})} - \theta^2\varepsilon^1, \end{cases} \tag{30}$$

where θ is the only design parameter of the observer, and $\varepsilon^1 = \hat{z}^1 - z^1$ represents the estimation error of the output.

The second technique uses the sliding mode principle [17] which is given, for the model of the pendulum, by:

$$\begin{aligned} \dot{\hat{z}}_1 &= \hat{z}_2 - \alpha_1\varepsilon_1 - \lambda_1 \operatorname{sgn}(\varepsilon^1), \\ \dot{\hat{z}}_2 &= \frac{f(\hat{z}) + g(\hat{z})u}{h(\hat{z})} - \alpha_2\varepsilon^1 - \lambda_2 \operatorname{sgn}(\varepsilon^1), \end{aligned} \tag{31}$$

where α_i and λ_i, $i = 1, 2$, are positive scalars, ε^1 is the estimation error of the state z^1, $f(\hat{z}), g(\hat{z})$ et $h(\hat{z})$ are, respectively, the estimation of the nonlinear functions $f(z), g(z)$ and $h(z)$ given by (25), and *sgn* is the usual signum function as defined in (17).

Finally, a sliding mode like output feedback controller with incorporation of a filtered integral action is then given by the following equations:

$$\begin{cases} u(\hat{z}) = \dfrac{h(\hat{z})}{g(\hat{z})}\left(\dot{z}_d^2 - \dfrac{f(z_d)}{h(z_d)} + \nu(\hat{e})\right), \\ \nu(\hat{e}) = -\dfrac{k\lambda^4}{\gamma}\tanh\left(a\left(\sigma^f + \dfrac{4}{\lambda}e^f + \dfrac{6\gamma}{\lambda^2}e^1 + \dfrac{4\gamma}{\lambda^3}\hat{e}^2\right)\right) \end{cases} \tag{32}$$

with :

$$\begin{cases} e^1 = z^1 - y_d \\ \hat{e}^2 = \hat{z}^2 - z_d^2 \end{cases},$$

Remark 2:

The separation principle still holds when we use a sliding mode observer. In fact, an important result proved in many papers related to output feedback control, in particular the works of Atassi and Khalil [24], [25], consists of the use a sufficiently fast high gain observer to estimate the missing states of the system together with a globally bounded stabilizing state feedback controller to recover closed loop system performance. They proved that the performance recovery includes recovery of exponential stability of the origin, the region of attraction and state trajectories. We can show that these conditions hold in our case. In fact, SMO belongs to HGO family, and one can make it sufficiently fast by a proper choice of its design parameters. On the other hand, the input of the system is done by the force u actuating on the translation of the cart and is limited by $\pm 10N$ [1]. So, the control input is globally bounded. Consequently, the separation theorem holds for our case. We can conclude that the origin ($z = 0$) is exponential stable and our goal is well realized (regulation of the position of the pendulum around the angle $z_1 = 0$ "upright position").

6.1 Simulation results

In order to illustrate the performance of the observer and the controller, numerical simulations were carried out by considering the following values of the parameters involved in the inverted pendulum model which are taken from [1]: $M = 1Kg, m = 0.2Kg, l = 0.6m$ and $g = 10ms^{-2}$. The "input gain" δ, is considered uncertain but constant and lies between $\delta \in [0.5, 2]$. This uncertainty could result from an uncertainty in the motor that provides the necessary force on the cart. We adopt the nominal value of this parameter which is equal to 1.

Simulation results were carried out to compare the performance of the controller based the two observers. In Fig. 1, we plotted the output of the system and the correspondent input control using a High Gain Observer (HGO) and a Sliding Mode Observer (SMO). The design parameters of the controller are given by: $\mathrm{k}=0.5$, $\mathrm{a}=10$, $\lambda=4$ and $\gamma=0.4$. Besides, the design parameter of the nonlinear observers are given by $\theta = 10$ for the HGO, and $\alpha_1 = 100$, $\alpha_2 = 20$, $\lambda_1 = 1$ and $\lambda_2 = 1$ for the SMO. In order to evaluate the robustness of the adopted controller against external disturbance, an output step like disturbance of magnitude $0.5\ rad$ is added to the output of the system over the interval $[4, 12]\sec$. The initial conditions were set to: $z^1(0) = z^2(0) = 0$, $\hat{z}^1(0) = 0.2$, and $\hat{z}^2(0) = 0$.

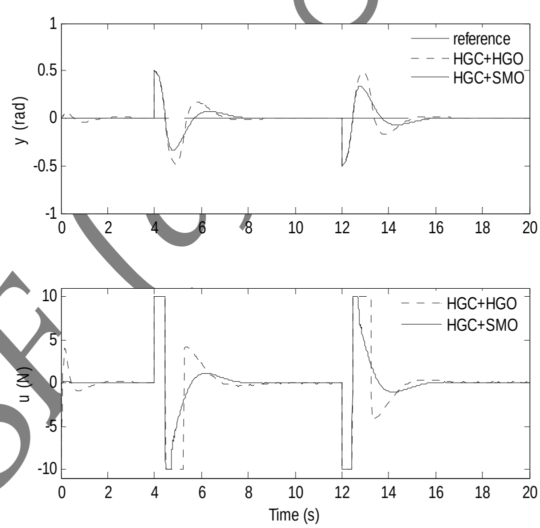

Fig. 1. Process output and input for disturbance rejection.

It can be shown, in Fig. 1 that the input control and the convergence of the output of the system to the reference signal using the High Gain Controller (HGC) based SMO is better than the one obtained using a HGO. Moreover, the control input signal resulting of the use of a SMO in the control loop is better than the case when we use a HGO. The superior rejection of the perturbation is due to the capability of the SMO to estimate the disturbances in a relatively small time as plotted in Fig. 2. Additionally, we adopt the Mean Square Error (MSE) as performance index to compare the two observation strategies. As shown in Table

1, the MSE of the resulting tracking error e^1 and the two states estimation error ε^1 and ε^2 issued from the use of a SMO is lower than values obtained with a HGO.

Table 1. MSE for step disturbances.

Variable	e^1	ε^1	ε^2
MSE (HGO)	0.0157	0.0028	0.058
MSE (SMO)	0.0085	$1.53*10^{-4}$	$2.12*10^{-4}$

Table 2. MSE for noisy measurements.

Variable	e^1	ε^1	ε^2
MSE (HGO)	0.0014	$6.52*10^{-4}$	0.0115
MSE (SMO)	0.0011	$7.35*10^{-5}$	$1.91*10^{-4}$

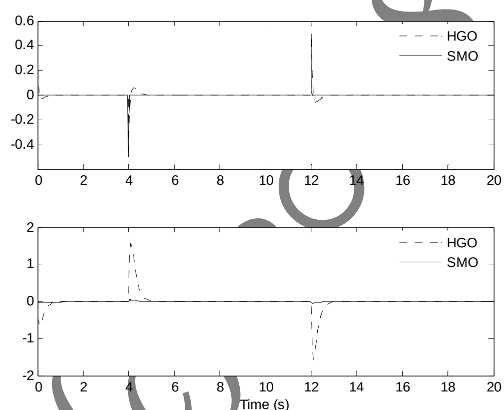

Fig. 2. Error estimation of the states z^1 and z^2.

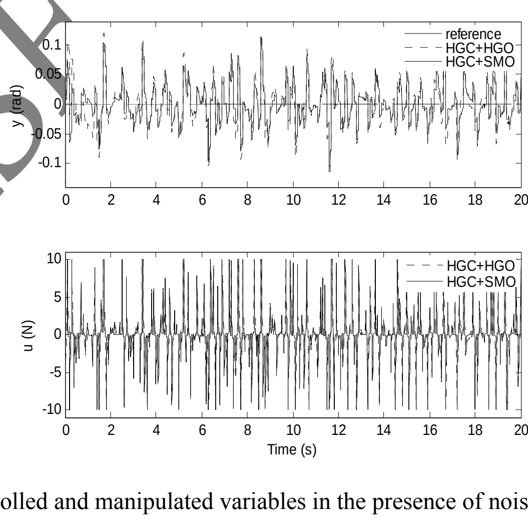

Fig. 3. Controlled and manipulated variables in the presence of noisy measurements.

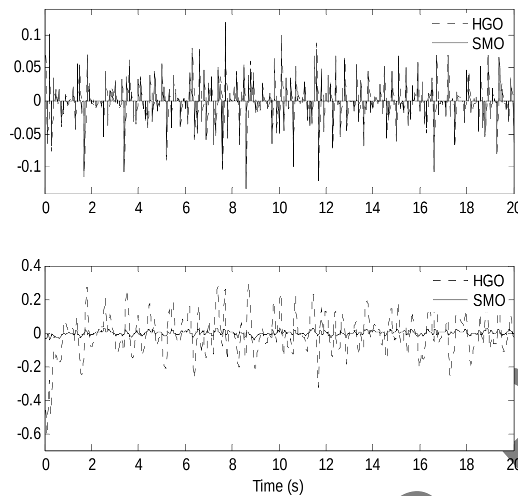

Fig. 4. Error estimation of the states z^1 and z^2 in the presence of noisy measurements.

To achieve practical conditions, and to check the robustness of both observers, the system output z^1 is assumed corrupted with uniformly distributed white noise signal of magnitude 10^{-4}. From the results plotted in Fig. 3, it is clear that the HGO is more influenced by the noise than the SMO. Besides, as shown in Fig. 4, the effectiveness of the use of a SMO appears on the states estimation error. In fact, showing the estimation error of the state z_2, we conclude that the SMO is less sensitive to noise measurement than the HGO. Results given in Table 2 confirm the less sensitivity of the SMO to stochastic noises. In fact, the MSE of the tracking error, and the two states estimation error issued from the use of a SMO are inferior than those obtained with a HGO.

Table 3. MSE for parametric uncertainty.

Variable	e^1	ε^1	ε^2
MSE (HGO)	0.0016	0.001	0.0153
MSE (SMO)	0.001	$1.1*10^{-4}$	$1.97*10^{-4}$

Remark 3: The use of a high value of the parameter θ in the HGO makes the estimation convergence faster, and so minimizes the tracking error. However, the system becomes more sensitive to stochastic noise and presents a high pick in the transient time. Thus, the choice of such parameter is based on a compromise between a good tracking of the state variations and a satisfactory dealing with noise rejection.

Now, we evaluate the robustness of the closed loop system against parametric uncertainties using state feedback control incorporating a filtered integral action based HGO and SMO. As the parameter δ is the only parameter which is unknown, we add 100% uncertainty in its nominal value, i.e. the value $\delta = 2$ is adopted in the development of the resulting control law which we did not change its design parameters. In Fig. 5, we plotted the simulation results for a regulation case when the high gain feedback controller is used in conjunction with each of two observers presented above. This plot shows that, when perturbations are added to the uncertain system, the SMO presents a superior disturbance rejection. This is due mainly to the fact that the SMO estimates better the perturbation than the HGO as we can see in Fig. 6. Additionally, the different values of the MSE reported in Table 3 prove the robustness and the effectiveness of the state feedback controller based SMO.

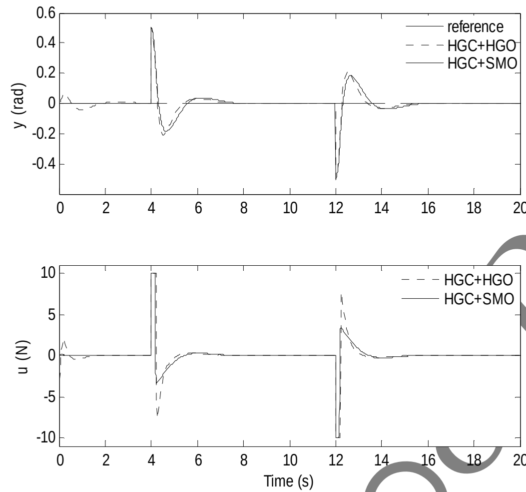

Fig. 5. Controlled and manipulated variables in the presence of parametric uncertainties.

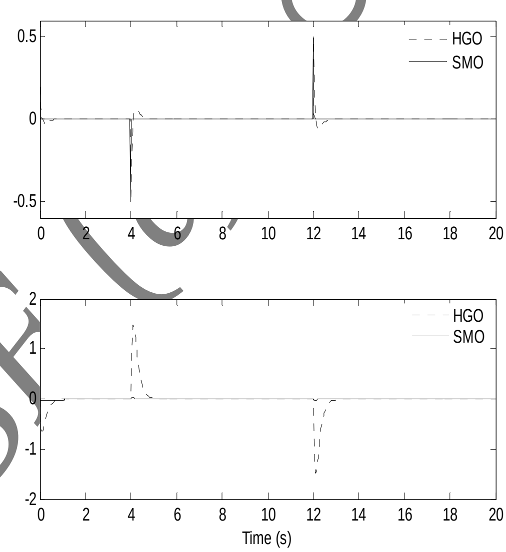

Fig. 6. Error estimation of the states z^1 and z^2 in the presence of parametric uncertainties.

7. CONCLUSION

This paper consists of the improvement and the application of a high gain state feedback control based nonlinear observers to deal with the problem of angular position regulation of an (unstable) inverted pendulum on a cart around an upright position. Into the output feedback control law, is incorporated a filtered integral action to carry out an asymptotic rejection of state and/or output step like disturbances and to reduce the system sensitivity against stochastic noise. We are interested in using a sliding mode observer instead of a high gain observer in the control law. We showed that with such observer the separation principle is verified. Moreover, the major advantage of the use of a sliding mode observer is that, besides the performance achieved under a state feedback controller, the resulting closed loop system can be made considerably more robust to external disturbances, noisy measurements and parametric uncertainties.

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