J. Automation & Systems Engineering

Regular paper

L^p-Stability Analysis of a Class of Nonlinear Fractional Differential Equations

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This paper investigates the L^p-stability properties of fractional nonlinear differential equations. Systems defined on a finite time interval are considered. The principal contributions are summarized in a theorem which gives sufficient conditions for bounded stability of fractional order systems. We show that the proposed results can not be extended to the case of systems defined on an infinite time interval.

Keywords: Fractional order derivative, Nonlinear fractional differential equations, Fractional order Systems, L^p-stability.

1. INTRODUCTION

The fractional calculus and fractional order differential equations attracted a great attention these last decades (see [14, 17, 10]). One of the most important reasons for this interest is their ability to model many natural systems and their seducing properties like robustness and dynamical behaviour. Fractional order systems have found many applications in various domains such as heat transfer, viscoelasticity, electrical circuit, electro-chemistry, dynamics, economics, polymer physics and control.

The study of stability for this kind of systems focuses a great interest in the research community. We can cite in this domain the works of Matignon [13] and Bonnet and Partington [2] for the stability of linear fractional systems, those of Khusainov [9], Bonnet and Partington [3], Chen and Moore [4] and Deng et al. [6] for fractional systems with time delay and Ladaci et al. [11] for fractional adaptive control systems. Ahn et al. have proposed robust stability test methods for fractional order systems [1,5]. Recently Lazarević [12] has studied the finite time stability of a fractional order controller for robotic time delay systems.

In this paper we are concerned by the stability analysis of fractional order systems represented by the following nonlinear differential equation:

$$D^{\alpha}x(t) = h\left(t, x(t)\right), \qquad x \in R^n,\ t \in R^+ \tag{1}$$

where $0 < \alpha < 1$, $h \in C\left(R^+ \times R^n, R^{n+}\right)$ is a continuous positive function, with the initial condition

$$D^{\alpha-1}x(t_0) = x_0 \tag{2}$$

In the following we will use the notation $h_x(t) = h\left(t, x(t)\right)$.

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This paper is organized as follows. In Section 2, we present some useful theoretical background. In section 3, the main result on L^p-stability of nonlinear fractional order systems defined on a finite interval is given with an illustrative example. Section 4 concludes the paper.

2. THEORETICAL BACKGROUND

The mathematical definition of fractional derivatives and integrals has been the subject of several different approaches [16]. In this paper we consider the following Riemann-Liouville definition [14],

Definition 1 (Fractional integral)

Let $\nu \in C$ such that $\mathrm{Re}(\nu) > 0$ and let g be piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval of $[0, \infty)$. Then for $0 \leq t_0 < t$ we call:

$$ {}_{t_0}D_t^{-\nu} g(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\nu-1} g(\tau) d\tau $$

the Riemann-Liouville fractional integral of g of order ν where $\Gamma(x) = \int\limits_0^\infty y^{x-1} e^{-y} dy$ is the Gamma function.

For simplicity we will note $D^\mu g(t)$ for ${}_{t_0}D_t^\mu g(t)$.

The Riemann-Liouville definition of fractional order derivative of g is now recalled.

Definition 2 (Fractional derivative)

Let g be a continuous function and let $\mu > 0$. Let m be the smallest integer that exceeds μ. The fractional derivative of g of order μ is defined as, $D^\mu g(t) = D^m \left[D^{-\nu} g(t) \right]$, $\mu > 0$ (if it exists) where $\nu = m - \mu > 0$.

Lemma 1

The solution of System (1)-(2) is given by the vector equality

$$ x(t) = \frac{x_0}{\Gamma(\alpha)} (t - t_0)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-\tau)^{\alpha-1} h_x(\tau) d\tau , $$

For the proof see for instance [7,8,15].

With no loss of generality we can take $x_0 = 0$ and $t_0 = 0$. This reduces our solution to,

$$ x(t) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-\tau)^{\alpha-1} h_x(\tau) d\tau \qquad (3) $$

Let us recall the definition of L^p-stability.

Definition 3

Let $1 \leq p \leq \infty$ and $\Omega \subset R^+$, the system (1) is $L^p(\Omega)$-stable if the solution $x(t)^T = (x_1, x_2, ..., x_n)$ defined by equation (3) belongs to $L^p(\Omega)$.

We now introduce the convolution product on $C([t_0,\infty),R)$ with $t_0 \geq 0$.

Definition 4

For all functions $f, g \in C(R^+, R)$ we define the operator $f \otimes g$ as follows:

$$f \otimes g(t) = \int_0^t f(t-\tau)g(\tau)d\tau, \qquad t \geq 0 \tag{4}$$

Lemma 2

The product $\otimes$ is a commutative internal composition rule on $C(R^+, R)$.

Proof. By using the change of variables $\tau = tu$, we obtain

$$f \otimes g(t) = \int_0^t f\big(t(1-u\big)\, g(tu)du$$

Due to the theorem about continuity of an integral depending on a parameter, we deduce that $t \mapsto f \otimes g(t)$ belongs to $C(R^+, R)$.

The commutativity property of the operator $\otimes$ can be easily proven by using the change of variables $u = t - \tau$ in (4).

In the next we show interesting properties of this product operator.

Lemma 3

Let $f, g \in C(R^+, R^+) \cap L^1([0,T])$ with $T > 0$, the following properties hold:

(i) $\displaystyle\int_0^T f \otimes g(t)dt = \int_0^T g(\tau)\left(\int_{t_0}^{T-\tau} f(t)dt\right)d\tau,$

(ii) $\displaystyle\left(\int_0^{\frac{T}{2}} f(t)dt\right)\left(\int_0^{\frac{T}{2}} g(\tau)d\tau\right) \leq \int_0^T f \otimes g(t)dt \leq \left(\int_0^T f(t)dt\right)\left(\int_0^T g(\tau)d\tau\right)$

(iii) For any $p \geq 1$ we have:

$$\left(\int_0^{\frac{T}{2}} g(\tau)^p d\tau\right)\left(\int_0^{\frac{T}{2}} f(t)dt\right)^p \leq \int_0^T (f \otimes g(t))^p dt \leq \left(\int_0^T g(\tau)^p d\tau\right)\left(\int_0^T f(t)dt\right)^p \tag{5}$$

(iv) Moreover, if $f, g \in L^1(R^+)$, the integral of $f \otimes g$ converges and we have:

$$\int_0^{+\infty} f \otimes g(t)dt = \left(\int_0^{+\infty} f(t)dt\right)\left(\int_0^{+\infty} g(\tau)d\tau\right)$$

(v) Moreover, if $f \in L^1(R^+)$ and $g \in L^p(R^+)$, we have for any $p \geq 1$:

$$\int_0^{+\infty} (f \otimes g(t))^p dt \leq \left(\int_0^{+\infty} f(t)dt\right)^p \left(\int_0^{+\infty} g(\tau)^p d\tau\right).$$

Proof.

(i)- By using the theorem of Fubini for positive functions, we have:

$$\int_0^T f\otimes g(t)dt=\int_0^T\int_0^t f(t-\tau)g(\tau)d\tau\ dt=\int_0^T\int_\tau^T f(t-\tau)g(\tau)dt\ d\tau$$

Then by using the change of variable $u=t-\tau$, we obtain:

$$\int_0^T f\otimes g(t)dt=\int_0^T g(\tau)\Big(\int_\tau^T f(t-\tau)d\tau\Big)dt=\int_0^T g(\tau)\Big(\int_0^{T-\tau} f(u)du\Big)dt \tag{6}$$

(ii)- Since f is positive, we have

$$\int_0^{T-\tau} f(t)dt\le\int_0^T f(t)dt$$

and the right inequality is immediate from (6). For the left inequality we remark that

$$\int_0^T g(\tau)\Big(\int_0^{T-\tau} f(t)dt\Big)d\tau\ge\int_0^{\frac{T}{2}} g(\tau)\Big(\int_0^{T-\tau} f(t)dt\Big)d\tau$$

Now if $0\le\tau\le\frac{T}{2}$, then $T-\tau\ge\frac{T}{2}$ and,

$$\int_0^T g(\tau)\Big(\int_0^{T-\tau} f(t)dt\Big)d\tau\ge\int_0^{\frac{T}{2}} g(\tau)\left(\int_0^{\frac{T}{2}} f(t)dt\right)d\tau\ge\int_0^{\frac{T}{2}} f(t)dt\int_0^{\frac{T}{2}} g(\tau)d\tau$$

(iii)- From (i) we have,

$$\int_0^T f\otimes g(t)dt=\int_0^T g(\tau)\Big(\int_0^{T-\tau} f(t)dt\Big)d\tau$$

then for the right inequality,

$$\begin{aligned}\int_0^T (f\otimes g(t))^p\,dt&=\int_0^T\Big\{g(\tau)\Big(\int_0^{T-\tau} f(t)dt\Big)\Big\}^p d\tau\\&\le\int_0^T g(\tau)^p\Big(\int_0^T f(t)dt\Big)^p d\tau\\&\le\Big(\int_0^T g(\tau)^p\,d\tau\Big)\Big(\int_0^T f(t)dt\Big)^p\end{aligned}$$

For the left inequality we use the same proof as in (ii). Since $0\le\tau\le\frac{T}{2}$, we have $T-\tau\ge\frac{T}{2}$ and:

$$\int_0^T\Big\{g(\tau)\Big(\int_0^{T-\tau} f(t)dt\Big)\Big\}^p d\tau\ge\left(\int_0^{\frac{T}{2}} g(\tau)^p\,d\tau\right)\left(\int_0^{\frac{T}{2}} f(t)dt\right)^p$$

(iv)- If T tends to infinity, then both of the right and the left sides of the inequality (5) converge to the same limit, that is:

$$\Big(\int_0^{+\infty} f(t)dt\Big)\Big(\int_0^{+\infty} g(\tau)d\tau\Big)$$

(v)- By using the preceding reasoning for the double inequality (5) we get,

$$\int_0^{+\infty} (f \otimes g(t))^p \, dt \le \left(\int_0^{+\infty} f(t)dt\right)^p \left(\int_0^{+\infty} g(\tau)^p \, d\tau\right).$$

3. MAIN RESULTS

The solution (3) of (1) can be rewritten using the product operator $\otimes$ defined in (4) as follows:

$$x(t) = \left(K_\varepsilon \otimes h_x\right)(t) \tag{7}$$

where K_α with $0 < \alpha < 1$ is the so called convolution kernel defined by

$$K_\alpha(u) = \frac{u^{\alpha-1}}{\Gamma(\alpha)} \tag{8}$$

Lemma 4

Consider the convolution kernel function K_α defined in (8), let $\varepsilon > 0$ then $K_\alpha \in L^p\left([0,\varepsilon]\right)$ if and only if:

$$\frac{p-1}{p} < \alpha < 1 \text{ and } p \ge 1$$

Proof.

We have

$$\int_0^\varepsilon \left(K_\alpha(u)\right)^p du = \int_0^\varepsilon \left(\frac{u^{\alpha-1}}{\Gamma(\alpha)}\right)^p du = \int_0^\varepsilon \frac{u^{(\alpha-1)p}}{\Gamma(\alpha)^p} du = \left[\frac{u^{(\alpha-1)p+1}}{\left((\alpha-1)p+1\right)\Gamma(\alpha)^p}\right]_0^\varepsilon \tag{9}$$

The generalized Riemann integral (9) is convergent if and only $p \ge 1$ and $(\alpha-1)p+1 > 0$ that is: $\frac{p-1}{p} < \alpha < 1$.

Then we have the main result on the L^p-stability of the solution of the system (1)-(2) defined on a finite time interval.

Theorem 1

Let $p \ge 1$ and consider the system defined by the fractional order differential equation (1)-(2) where the time $t \in \left[0, t_f\right]$ then the system (1)-(2) is $L^p\left(\left[0, t_f\right]\right)$-stable if and only if:

$$\frac{p-1}{p} < \alpha < 1 \text{ and } h_x \in L^1\left(\left[0, t_f\right]\right).$$

Proof.

As the operator $\otimes$ is commutative, we have:

$${}_0^{t_f}\left\|x(t)\right\|_p = \left(\int_0^{t_f} (K_\alpha \otimes h_x)^p \, dt\right)^{1/p} = \left(\int_0^{t_f} (h_x \otimes K_\alpha)^p \, dt\right)^{1/p}$$

and again from Lemma 3-(iii) and equality (7) we get:

$$ {}_{0}^{t_f}\|x(t)\|_p \leq \left(\int_0^{t_f} h_x(\tau)d\tau\right)\left(\int_0^{t_f} (K_\alpha(t))^p\, dt\right)^{1/p} \tag{10} $$

From Lemma 4, we have that $\int_0^{t_f} K_\alpha(t)^p\, dt < +\infty$ if and only if $\frac{p-1}{p} < \alpha < 1$.

If in addition $h_x \in L^1\left(\left[0, t_f\right]\right)$, equation (10) implies that

$$ {}_{0}^{t_f}\|x(t)\|_p \leq \frac{t_f^{\frac{(\alpha-1)p+1}{p}}}{((\alpha-1)p+1)^{\frac{1}{p}}\,\Gamma(\alpha)} \int_0^{t_f} h_x(\tau)d\tau < +\infty $$

Example 1

Let us consider the system

$$ D^{\frac{3}{4}}x(t) = \frac{B(x)}{(t+1)^3}, \qquad 0 \leq t \leq 1 $$

where $B(x)$ is a positive real function bounded by a constant $c > 0$ as for instance the function $x \mapsto |\sin x|$.

Then, by using Theorem 9 with $p = 2$ and $\alpha = \frac{3}{4}$, we deduce that:

$$ {}_{0}^{1}\|x(t)\|_2 \leq \frac{3\sqrt{2}c}{8\Gamma\left(\frac{3}{4}\right)} $$

Remark 1

The generalization to the infinite case where $t \in R^+$ is not possible because the kernel function $u \mapsto K_\alpha(u)$ does not belong to $L^1\left(R^+\right)$. Thus we can not use points (iv) and (v) of lemma 3.

Even if the system is defined since $t_0 > 0$, the generalization to the infinite case where $t \in \left[t_0, +\infty\right)$ is not possible. Indeed the solution of System (1)-(2) given by:

$$ x(t) = \int_{t_0}^{t} K_\alpha(t-\tau)h_x(\tau)\, d\tau, \qquad t \geq t_0 > 0 \tag{11} $$

can not be defined by using a convolution product which is commutative. So, Theorem 1 can not be extended to the case where $t \in \left[t_0, +\infty\right)$ with $t_0 > 0$.

A commutative convolution product with initial condition $t_0 > 0$ which can be used would be:

$$ f \otimes g(t) = \int_{t_0}^{t} f(t + t_0 - \tau)g(\tau)d\tau, \qquad t \geq 0 $$

However, our kernel function $K_\alpha(t-\tau)$ as it appears in the definition of the solution of System (1)-(2) given by Equation (11) does not depend on the initial time t_0.

4. CONCLUSION

In this paper the L^p-stability properties of fractional nonlinear differential equations are considered. Sufficient conditions for L^p-stability of the fractional order system are presented in the case of finite time window, with analytical proofs based on the properties of a special convolution product. This work opens a new method for the L^p-stability analysis of a class of nonlinear fractional order systems.

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