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High gain observer design for a class of MIMO non uniformly observable uncertain systems

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Abstract-A high gain observer is proposed for a class of MIMO non-uniformly observable systems including uncertainties. The gain of the proposed observer is issued from the resolution of a Lyapunov differential equation and its tuning can be achieved through the choice of a single scalar parameter. The observer design is made under the assumption that the considered inputs satisfy certain persistent excitation conditions. It is shown that in the absence of uncertainties, the observation error converges exponentially to zero. In the case where the uncertainties are bounded, this error can be made as small as desired by choosing high values of the observer design parameter. A simulation example is given in order to highlight the observer performance.

Keywords: Nonlinear systems, High gain observer, non uniform observability, Lyapunov differential equation.

1. INTRODUCTION

In spite of the intensive research efforts throughout the last twenty years, the observer design for MIMO nonlinear systems is still an open problem. Various approaches have been proposed to design state observers for different classes of nonlinear systems (see for instance [1,2,3,6,11,12,13,15,16,18,22,25,26] and references therein) but none of them provides a general solution as in the linear time invariant case. The seminal approaches dealing with observers design for nonlinear systems are based on appropriate coordinates transformation which lead to linear error dynamics up to an output injection allowing thereby to design a Luenberger or Kalman observer (see for instance [5,14,17,20,21,23,24,27]. An intensive research activity has been devoted to the class of systems that are observable for any input, i.e. uniformly observable systems (see for instance [15, 16, 19]. In [15], the authors proposed a canonical form for single output nonlinear systems in the control affine case. This canonical form is composed of a fixed linear dynamics together with a nonlinear triangular controlled one. The extension of the observer design has been achieved in [16] to deal with the non-control affine case and in [19] for a class of MIMO uniformly observable systems characterized by a normal form where the nonlinearities are also triangular.

Several extensions of the observer design have been proposed for some particular classes of MIMO uniformly observable systems. The underlying observers are characterized by a constant gain which is often issued from an algebraic Lyapunov equation. In the case of non-uniformly observable systems, there is no a systematic approach to deal with the observer design since these systems may admit inputs that render them unobservable. The available contributions are rather genuine extensions of the observer design approach adopted for uniformly observable systems up to some sufficient conditions on the system inputs which allow the systems to be sufficiently observable to perform an appropriate observer design. These conditions are generally referred to as persistent excitation conditions which require the positive definiteness of the system observability Gramian. In [4], the authors introduced the notion of local regular inputs and gave sufficient conditions which allow the characterization of systems that can be immersed under higher dimension normal form composed of an affine part depending on the input and output and a triangular controlled nonlinear part. A high gain observer involving a Lyapunov ODE has then been designed on the basis of the normal form. The class of systems considered in [4] has been revisited in [8] where the authors introduced the notion of regular inputs allowing thereby to weaken the involved persistent excitation condition and to design a high gain observer.

In this paper, we shall design a high gain observer for a class of uncertain systems including those considered in [4] and [8]. The gain of the observer is issued from the resolution of a Lyapunov differential equation and its convergence is achieved under a set of conditions, e.g. on the system inputs. The underlying persistent excitation condition is similar to that given in [4] and [8] but can be easily checked on-line. It is shown that in the absence of uncertainties, the observation error converges exponentially to zero. In the case where the uncertainties are bounded, this error can be made as small as desired by choosing high values of the observer design parameter.

The paper is organized as follows. Section 2 is devoted to the problem formulation with a particular emphasis on the considered class of systems. Section 3 is devoted to the observer design. The observer equations are derived and the assumptions required to guarantee its convergence are given and discussed. It is worth mentioning that the observer gain is issued from the resolution of a Lyapunov ODE and its calibration can be achieved through the tuning of a single real parameter. The convergence analysis of the observation error is detailed in section 4. The boundedness of the solution of the Lyapunov ODE from which the gain of the observer is provided, is first established. Then, an upper bound for the observation error is derived. A particular attention is paid to the exponential convergence of the proposed observer in the absence of uncertainties and to the fact that the asymptotic observation error can be made as small as desired for high values of the observer design parameter. Simulation results are given in section 5 for illustration purposes. Finally, some concluding remarks are given in section 6.

Throughout the paper, for any positive integers k and l, I_k and 0_k denote the k-dimensional identity and null matrices respectively, $0_{k,l}$ denotes the $k\times l$ rectangular null matrix, $\|\cdot\|$ denotes the euclidian norm and for any Symmetric Positive Definite (SPD) time-varying matrix $Q(t)$, $\lambda_M(Q(t))$ (resp. $\lambda_m(Q(t))$) will be used to denote the largest (resp. The smallest) eigenvalue of $Q(t)$ and $\overline{\lambda}_M(Q) = \sup_{t\geq t_o} \lambda_M(Q(t))$, $\underline{\lambda}_m(Q) = \min_{t\geq t_o} \lambda_m(Q(t))$ where t_o is any fixed negative real.

2. PROBLEM FORMULATION

Consider the following class of MIMO dynamical systems governed by the following state space representation:

$$\begin{cases} \dot{x}(t) = F(u,x)x(t) + \varphi(u,x) + B\varepsilon(t) \\ y(t) = C(u)x(t) \end{cases} \tag{1}$$

$$F(u,x) = \begin{pmatrix} 0 & F_1(u,x) & 0 & \dots & \dots & 0 \\ 0 & 0 & F_2(u,x) & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \dots & & \dots & 0 & F_{q-1}(u,x) \\ 0 & \dots & & & \dots & 0 \end{pmatrix},$$

$$C(u) = \begin{pmatrix} F_o(u) & 0 & \dots & 0 \end{pmatrix} \text{ and } B^T = -\begin{pmatrix} 0 & \dots & 0 & I_{n_q} \end{pmatrix} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in U$ a compact set of $\mathbb{R}^s$ and $y \in R^p$ are respectively the state, the input and the output of the system with $x(t) = \left(x^{1T} \dots x^{qT}\right)^T \in \mathbb{R}^n$ where $x^{(k)} \in \mathbb{R}^{n_k}$ for $k \in [1,q]$, with $n_1 = p$ and $\sum_{k=1}^{q} n_k = n$, $F_o(u)$ is $n_o \times n_1$ matrix with $n_o = n_1 = p$ and each $F_k(u,x)$ is a $n_k \times n_{k+1}$ matrix which is triangular with respect to x, i.e. $F_k(u,x) = F(u,x^1,\dots,x^k)$ for $k \in [1,q-1]$, φ is a nonlinear vector function that has a triangular structure with respect to x, i.e. $\varphi(u,x) = \left(\varphi^{1T}(u,x^1)\, \varphi^{2T}(u,x^1,x^2) \dots \varphi^{qT}(u,x)\right)^T$; $\varepsilon \in R^{n_q}$ is an unknown function.

In the case where each matrix F_k is column full rank, system (1) belongs to the class of systems considered in [19] which characterizes a subclass of systems which are observable for any input and for which the authors proposed a high gain observer where the gain is issued from an algebraic Lyapunov equation. Note that the fact that F_k is of full rank implies that $n_k \geq n_{k+1}$ for $k \in [1,q-1]$. In [7], the authors considered a similar class of systems where the matrices F_k are reduced to positive bounded real-valued functions and proposed a high gain observer with a time-varying gain issued from the resolution a Riccatti ODE. In [4], the considered class of systems is similar to systems (1) but the functions F_k only depend on the inputs and outputs. The authors gave sufficient conditions that characterize the class of systems that can be immersed under form (1). A high gain observer with a gain issued from a Lyapunov ODE has also been proposed for system (1). The class of systems considered in [4] has been revisited in [8] where the authors proposed a design of a high gain observer under a relatively relaxed persistent excitation condition.

A high gain observer design shall be proposed to estimate the state of system (1).

3. THE STATE OBSERVER DESIGN

We consider the following assumptions which are mainly required for high gain observer design.

A1 The state $x(t)$ and the control $u(t)$ are bounded, i.e. $x(t) \in X$ and $u(t) \in U$ where $X \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^s$ are compacts sets.

A2 The functions F_k and φ are Lipschitz with respect to x uniformly in u where $(u,x) \in U \times X$. The corresponding Lipschitz constants will be denoted by L_F and L_φ, respectively.

A3 The unknown function ε is essentially bounded i.e.

$$\exists \delta_\varepsilon > 0 \; ; \sup_{t \geq 0} Ess \, \| \varepsilon(t) \| \leq \delta_\varepsilon \tag{3}$$

Since the state is confined to the bounded set X, one can assume that Lipschitz prolongations of the nonlinearities, using smooth saturation functions, have been carried out and that the functions F_k and φ are provided from these prolongations. This allows to conclude that for any bounded input $u \in U$, the functions F_k and φ are globally Lipschitz with respect to x and are bounded for all $x \in \mathbb{R}^n$ (see [10] and references therein for more details). We shall denote throughout this paper by x_M the upper bound of x i.e.

$$x_M = \sup_{t \geq 0} \| x(t) \| \tag{4}$$

Before providing our candidate observer, let us introduce some useful notations and definitions related to high gain observer design.

- Let Δ_θ be the (block) diagonal matrix defined by:

$$\Delta_\theta = diag\left[I_{n_1}, 1/\theta I_{n_2}, \ldots, 1/\theta^{q-1} I_{n_q} \right] \tag{5}$$

where θ is a positive scalar. One can easily check the following equalities:

$$\begin{aligned} \Delta_\theta F(u,x) \Delta_\theta^{-1} &= \theta F(u,x), \\ C(u)\Delta_\theta^{-1} &= C(u)\Delta_\theta = C(u) \end{aligned} \tag{6}$$

Now, let us consider the following dynamical system:

$$\dot{\hat{x}} = F(u,\hat{x})\hat{x} + \varphi(u,\hat{x}) - \theta \Delta_\theta^{-1} S^{-1}(t) C^T(u)(C(u)\hat{x} - y) \tag{7}$$

where $\hat{x} = (\hat{x}^{1T} \ldots \hat{x}^{qT})^T \in \mathbb{R}^n$ with $\hat{x}^k \in \mathbb{R}^{n_k}$, u and y are respectively the input and the output of system (1) and S is a square symmetric matrix governed by the following Lyapunov ODE:

$$\dot{S}(t) = -\theta\left(S(t) + F^T(u,\hat{x})S(t) + S(t)F(u,\hat{x}) - C^T(u)C(u)\right) \qquad (8)$$

with $S(0) = S^T(0) > 0$ and $\theta > 0$ is a scalar design parameter.

Let $\Phi_{u,\hat{x}}(t,s)$ be the state transition matrix of the state affine system:

$$\dot{\xi}(t) = F(u(t),\hat{x}(t))\xi(t) \qquad (9)$$

where the state $\xi \in \mathbb{R}^n$ and $\hat{x}$ are the input of the system and they respectively correspond to the input of system (1) and to the state of the dynamical system (7); the matrix F is defined as in (2). Recall that the matrix $\Phi_{u,\hat{x}}(t,s)$ is defined as follows: $\begin{cases} d\left(\Phi_{u,\hat{x}}(t,s)\right)/dt & = & F(u(t),\hat{x}(t))\Phi_{u,\hat{x}}(t,s) \\ \Phi_{u,\hat{x}}(s,s) & = & I_n \end{cases}$

(10)

This transition matrix allows to introduce the following additional hypothesis which is crucial for the observer convergence analysis:

A4 The input u is such that for any trajectory $\hat{x}(0) \in X$ of system (7) starting from $\hat{x}(0) \in X$, the following persistent excitation condition is

$$\exists\theta^\star > 0\ ;\exists\delta_o > 0;\forall\theta \geq \theta^\star;\forall t \geq \frac{1}{\theta},$$

satisfied:

$$\text{one has } \int_{t-\frac{1}{\theta}}^{t} \Phi^T_{u,\hat{x}}(s,t)C^T(u(s))C(u(s))\Phi_{u,\hat{x}}(s,t)ds \geq \frac{\delta_o}{\theta\alpha(\theta)}\Delta_\theta^2$$

(11) where $\alpha(\theta)$ is a positive function satisfying

$$\forall\theta > 0, \alpha(\theta) \geq 1 \text{ and } \lim_{\theta\to\infty}\frac{\alpha(\theta)}{\theta^2} = 0 \qquad (12)$$

Remark:

The persistent excitation condition involved in assumption **A4** is similar to that given in [2] and [4] where the authors introduced the notion of local regular inputs, i.e. the inputs for which inequality (11) is satisfied with $\alpha(\theta) \equiv 1$. In [8], the authors introduced the notion of regular inputs and showed that the class of regular inputs includes that of local regular inputs. Using the same arguments as in [8], one can show that the class of inputs satisfying assumption **A3** includes all local regular inputs but is included in the class of regular inputs.

The dynamical system (7) is actually a state observer for system (1) as pointed out by the following result.

Theorem 1:

Consider system (1) subject to assumptions **A1** to **A3**. Then, for any input satisfying assumption **A4**, one has:

Then,

$$\forall\rho>0;\exists\eta>0;\forall\theta\geq max(1,\theta^{\star});\forall u\, s.t.\|u\|_{\infty}\leq\rho;\forall\hat{x}(0)\in\mathbb{R}^{n};$$

$$\text{we have}\ \|\hat{x}(t)-x(t)\|\leq\eta\sqrt{\alpha(\theta)}\theta^{q-1}e^{-\mu_\theta\theta t}\|\hat{x}(0)-x(0)\|+\sqrt{\frac{\alpha(\theta)}{\theta^2}}\frac{\eta}{\mu_\theta}\delta_\varepsilon$$

where x is the unknown system trajectory associated to the input u, $\hat{x}$ is any trajectory of the observer (7) associated to (u,y), $\theta^{\star}$ and $\alpha(\theta)$ are given by Assumption **A4**, δ_ε is the upper essential bounds of $\xi(t)$ as given by (3) and finally μ_θ is a positive real-valued function of θ and is such that $\lim_{\theta\to\infty}\mu_\theta=1$

Remark:

Notice that in the absence of uncertainties i.e. $\delta_\varepsilon=0$, the observation error converges exponentially to zero. In the case where $\delta_\varepsilon\neq0$, the observation error is ultimately bounded and the ultimate can be made as small as desired since $\lim_{\theta\to\infty}\frac{\alpha(\theta)}{\theta^2}=0$

3. PROOF OF THEOREM 1:

We shall first show that the matrix S governed by (8) is SPD and we shall derive a lower bound for its smallest eigenvalue. Indeed, from (6), one can show that the transition matrix, $\tilde{\Phi}_{u,\hat{x}}$, of the following state affine system

$$\dot{\xi}(t)=\theta F(u(t),\hat{x}(t))\xi(t) \tag{13}$$

can be explained as follows

$$\tilde{\Phi}_{u,\hat{x}}(t,s)=\Delta_\theta\Phi_{u,\hat{x}}(t,s)\Delta_\theta^{-1} \tag{14}$$

where $\Phi_{u,\hat{x}}$ is defined by (10).

As a result, the matrix $S(t)$, solution of the ODE (8), can be expressed as follows:

$$S(t) = e^{-\theta t}\tilde{\Phi}_{u,\hat{x}}^{T}(0,t)S(0)\tilde{\Phi}_{u,\hat{x}}(0,t) + \theta\int_{o}^{t} e^{-\theta(t-s)}\tilde{\Phi}_{u,\hat{x}}^{T}(s,t)C^{T}(u(s))C(u(s))\tilde{\Phi}_{u,\hat{x}}(s,t)ds$$
$$= e^{-\theta t}\Delta_{\theta}^{-1}\Phi_{u,\hat{x}}^{T}(0,t)\Delta_{\theta}S(0)\Delta_{\theta}\Phi_{u,\hat{x}}(0,t)\Delta_{\theta}^{-1}$$
$$+\theta\int_{o}^{t} e^{-\theta(t-s)}\Delta_{\theta}^{-1}\Phi_{u,\hat{x}}^{T}(s,t)\Delta_{\theta}C^{T}(u(s))C(u(s))\Delta_{\theta}\Phi_{u,\hat{x}}(s,t)\Delta_{\theta}^{-1}ds$$

And using the facts that $C(u)\Delta_{\theta} = C(u)$ and $S(0)$ is SPD, one gets for $t \geq \frac{1}{\theta}$:

$$S(t) \geq \theta\int_{o}^{t} e^{-\theta(t-s)}\Delta_{\theta}^{-1}\Phi_{u,\hat{x}}^{T}(s,t)C^{T}(u(s))C(u(s))\Phi_{u,\hat{x}}(s,t)\Delta_{\theta}^{-1}ds$$
$$\geq \theta\int_{t-\frac{1}{\theta}}^{t} e^{-\theta(t-s)}\Delta_{\theta}^{-1}\Phi_{u,\hat{x}}^{T}(s,t)C^{T}(u(s))C(u(s))\Phi_{u,\hat{x}}(s,t)\Delta_{\theta}^{-1}ds \qquad (15)$$
$$\geq \theta e^{-1}\int_{t-\frac{1}{\theta}}^{t} \Delta_{\theta}^{-1}\Phi_{u,\hat{x}}^{T}(s,t)C^{T}(u(s))C(u(s))\Phi_{u,\hat{x}}(s,t)\Delta_{\theta}^{-1}ds\,\frac{\delta_{o}e^{-1}}{\alpha(\theta)}I_{n}$$

Where δ_o and $\alpha(\theta)$ are given by (11) and (12), respectively. According to inequality (15), one clearly has

$$\underline{\lambda}_{m}(S) \geq \frac{\delta_{o}e^{-1}}{\alpha(\theta)} \qquad (16)$$

We shall now show that $\overline{\lambda}_{M}(S)$ is bounded and its upper bound is independent of θ. To this end, we shall show that this property is satisfied for each entry of the matrix $S(t)$. Indeed, let us denote by S_{ij} the (block) entry of the matrix S located at the row i and the column j. Then, according to the ODE (8), one has:

$$\dot{S}_{11} = -\theta(S_{11} - F_{o}^{T}(u)F_{o}(u)) \qquad (17)$$

$$\dot{S}_{1j} = -\theta\left(S_{1j} + S_{1,j-1}F_{j-1}(u,\hat{x})\right) \, for \, j \in [2,q] \qquad (18)$$

$$\dot{S}_{ij} = -\theta\left(S_{ij} + S_{i,j-1}F_{j-1}(u,\hat{x}) + F_{i-1}^{T}(u,\hat{x})S_{i-1,j}\right) for \, i \in [2,q] \, and \, j \in [i,q] \qquad (19)$$

Recall that according to hypotheses **A1** and **A2**, the matrices F_k for $k \in [0, q-1]$ are bounded. Indeed, set $F_M = \max(\sup_{t\geq 0} \| F_o(u(t)) \|, \sup_{t\geq 0} \| F_k(u(t), \hat{x}(t)) \|)$.

According to (17) one has

$$\begin{aligned}\| S_{11}(t) \| &\leq e^{-\theta t} \| S_{11}(0) \| + \theta \int_o^t e^{-\theta(t-s)} \| F_o^T(u(s)) F_o(u(s)) \| ds \\ &\leq \| S_{11}(0) \| + F_M^2 \theta \int_o^t e^{-\theta(t-s)} ds \\ &= \| S_{11}(0) \| + F_M^2 (1 - e^{-\theta t}) \\ &\leq \| S_{11}(0) \| + F_M^2\end{aligned}$$

Now, for $j \geq 2$, one shall proceed by induction on j in order to show that S_{1j} is bounded with a bound that does not depend on θ. Indeed assume that $S_{1,j-1}$ is bounded and let us denote by $S_M = \sup_{t \geq 0} \| S_{1,j-1}(t) \|$, one gets for $j \in [2, q]$:

$$\begin{aligned}\| S_{1j}(t) \| &\leq e^{-\theta t} \| S_{1j}(0) \| + \theta \int_o^t e^{-\theta(t-s)} \| S_{1,j-1}(s) F_{j-1}(u, \hat{x}) \| ds \\ &\leq \| S_{1j}(0) \| + F_M S_M\end{aligned} \qquad (20)$$

At this step, we showed that all the entries located at the first row (and the first column) of the symmetric matrix S are bounded. We shall proceed by induction on the row number i in order to show that all the entries of a row are bounded. Indeed, suppose that all the entries of the rows 1 to $i-1$ are bounded (with a bound that does not depend on θ) and let us show that all the entries of the row i are also bounded. Since S is symmetric, all the entries $S_{i,j}$ with $i < j$ are bounded by the induction hypothesis. In particular, one has $S_{i-1,i} = S_{i,i-1}^T$ and these matrices are bounded with an upper bound, e.g. S_M, independent of θ. Now, we shall show that $S_{i,j}$ is bounded for $j > i$. Let us first consider the case where $j = i$. Indeed, according to (18), one has

$$\begin{aligned}\| S_{ii}(t) \| &\leq e^{-\theta t} \| S_{ii}(0) \| + \theta \int_o^t e^{-\theta(t-s)} \| S_{i,i-1} F_{j-1}(u, \hat{x}) + F_{i-1}^T(u, \hat{x}) S_{i-1,i} \| ds \\ &\leq \| S_{ii}(0) \| + 2 F_M S_M\end{aligned}$$

Now, let $j > i$ and assume that $S_{i,j-1}$ is bounded; let us show that $S_{i,j}$ is also bounded. Indeed, according to the induction hypothesis, the entry $S_{i-1,j}$ located at the row $i-1$ is bounded. And from equation (18), one gets

$$\begin{aligned}\| S_{ij}(t) \| &\leq e^{-\theta t} \| S_{ij}(0) \| + \theta \int_o^t e^{-\theta(t-s)} \| S_{i,j-1} F_{j-1}(u, \hat{x}) \\ &\quad + F_{i-1}^T(u, \hat{x}) S_{i-1,j} \| ds \\ &\leq \| S_{ij}(0) \| + 2 F_M S_M\end{aligned}$$

To summarize, all the entries of the matrix S are bounded with an upper bound independent of θ. As a result, the largest eigenvalue of S , i.e. $\overline{\lambda}_M(S)$, is also independent of θ.

Let us now provide an upper bound for the observation error. To this end, set $\overline{x} = \Delta_\theta \tilde{x}$ where $\tilde{x}(t) = \hat{x} - x$ is the observation error. Using the equalities (6), one gets

$$\begin{aligned} \dot{\overline{x}} &= \theta(F(u,\hat{x}) - S^{-1}C^T(u)C(u))\overline{x} + \Delta_\theta\left(\tilde{F}(u,\hat{x},x)x + \tilde{\varphi}(u,\hat{x},x)\right) - \Delta_\theta B\varepsilon(t) \\ &= \theta(F(u,\hat{x}) - S^{-1}C^T(u)C(u))\overline{x} + \Delta_\theta\left(\tilde{F}(u,\hat{x},x)x + \tilde{\varphi}(u,\hat{x},x)\right) - \frac{1}{\theta^{q-1}} \end{aligned} \tag{21}$$

Let $V(t,\overline{x}(t)) = \overline{x}^T(t)S(t)\overline{x}(t)$ be the candidate Lyapunov function. Using (8), one gets:

$$\begin{aligned} \dot{V} &= \overline{x}^T \dot{S}\overline{x} + 2\overline{x}^T S\dot{\overline{x}} \\ &= -\theta V - \theta\overline{x}^T C^T(u)C(u)\overline{x} + 2\overline{x}^T S\Delta_\theta\left(\tilde{F}(u,\hat{x},x)x + \tilde{\varphi}(u,\hat{x},x)\right) \\ &\quad - 2\overline{x}^T SB\frac{\varepsilon}{\theta^{q-1}} \end{aligned} \tag{22}$$

And proceeding as in [9, 10] one can show that for $\theta \geq 1$

$$\begin{aligned} 2\overline{x}^T S\Delta_\theta \tilde{F}(u,\hat{x},x)x &\leq 2\sqrt{n}\sqrt{\overline{\lambda}_M(S)}\sqrt{V(t,\overline{x})}L_F x_M \,\|\overline{x}\| \\ &\leq 2\sqrt{n}\,\sigma_S L_F x_M V(t,\overline{x}) \\ 2\overline{x}^T S\Delta_\theta \tilde{\varphi}(u,\hat{x},x) &\leq 2\sqrt{n}\sqrt{\overline{\lambda}_M(S)}\sqrt{V(t,\overline{x})}L_\varphi \,\|\overline{x}\| \\ &\leq 2\sqrt{n}\,\sigma_S L_\varphi V(t,\overline{x}) \\ 2\overline{x}^T S\varepsilon(t) &\leq 2\sqrt{\overline{\lambda}_M(S)}\sqrt{V(t,\overline{x})}\,\|\varepsilon(t)\| \end{aligned}$$

Where $\sigma_S = \sqrt{\overline{\lambda}_M(S)/\underline{\lambda}_m(S)}$ and L_F , L_φ are the respective Lipschitz constants of F and φ as stated in assumption **A2**; x_M is the upper bound of x given by (4). According to (16), one has

$$\sigma_S \leq \sqrt{\alpha(\theta)}\sqrt{\frac{\overline{\lambda}_M(S)e}{\delta_o}} \tag{23}$$

Combining (22) with (23), one gets

$$\dot{V} \le -\theta V + 2\sqrt{\alpha(\theta)}\sqrt{\frac{n\bar{\lambda}_M(S)e}{\delta_o}}(L_F x_M + L_\varphi)V$$
$$+2\sqrt{\bar{\lambda}_M(S)}\sqrt{V(t,\bar{x})}\,\|\,\varepsilon(t)\,\| \tag{24}$$
$$\le -2\theta\mu_\theta V + 2\frac{\sqrt{\bar{\lambda}_M(S)}}{\theta^{q-1}}\sqrt{V(t,\bar{x})}\,\|\,\varepsilon(t)\,\|$$

where $\mu_\theta = \left(\frac{1}{2} - \sqrt{\frac{\alpha(\theta)}{\theta^2}}\sqrt{\frac{n\bar{\lambda}_M(S)e}{\delta_o}}(L_F x_M + L_\varphi)\right)$. Notice that according to the property of α given by (12), one has $\lim_{\theta\to\infty}\mu_\theta = 1$ as stated in Theorem 1.

Inequation (24) can be rewritten as follows:

$$\frac{d\sqrt{V}}{dt} \le -\theta\mu_\theta\sqrt{V} + \frac{\sqrt{\bar{\lambda}_M(S)}}{\theta^{q-1}}\,\|\,\varepsilon(t)\,\| \tag{25}$$

Integrating the above equation from 0 to $t > 0$ gives:

$$\sqrt{V(t,\bar{x}(t))} \le e^{-\theta\mu_\theta t}\sqrt{V(0,\bar{x}(0))} + \frac{\sqrt{\bar{\lambda}_M(S)}}{\theta^q\mu_\theta}\delta_\varepsilon \tag{26}$$

Coming back to the original coordinates and from the fact that $\|\,\bar{x}\,\| \le \|\,\tilde{x}\,\| \le \theta^{q-1}\,\|\,\bar{x}\,\|$, one gets:

$$\|\,\bar{x}(t)\,\| \le \sigma_S\theta^{q-1}e^{-\theta\mu_\theta t}\,\|\,\bar{x}(t)\,\| + \frac{\sigma_S}{\theta\mu_\theta}\delta_\varepsilon \tag{27}$$

Finally, according to (23), one gets:

$$\|\,\bar{x}(t)\,\| \le \sqrt{\frac{\bar{\lambda}_M(S)e\alpha(\theta)}{\delta_o}}\theta^{q-1}e^{-\theta\mu_\theta t}\,\|\,\bar{x}(t)\,\|$$
$$+\sqrt{\frac{\alpha(\theta)}{\theta^2}}\sqrt{\frac{\bar{\lambda}_M(S)e}{\delta_o\mu_\theta}}\delta_\varepsilon \tag{28}$$

The constant η required by the theorem is $\eta = \sqrt{\frac{\bar{\lambda}_M(S)e}{\delta_o}}$. This ends the proof.

3. EXAMPLE

Consider the following dynamical system:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} u_1 x_1 x_2 + u_2 x_1^2 x_3 + x_1 u_3 \\ x_3 - 10x_2 + u_3 - x_2^3 + \varepsilon_1(t) \\ -x_3 - 2x_1 x_2 - 10u_1 x_2 \end{bmatrix} \\ y(t) = x_1(t) \end{cases} \tag{29}$$

Where $x_i, u_j \in \mathbb{R}$ $i = 1,...,3$ $j = 1,...,3$. System (29) is under form

$q = 2, x = \begin{bmatrix} x^{(1)T} & x^{(2)T} \end{bmatrix}^T, x^{(1)} = \begin{bmatrix} x_1 \end{bmatrix}, x^{(2)} = \begin{bmatrix} x_2 & x_3 \end{bmatrix}^T,$

$F_1(u, x^1) = \begin{bmatrix} u_1 x_1 & u_2 x_1^2 \end{bmatrix}, \varphi(x(t), u(t)) = \begin{bmatrix} x_1 u_3 \\ x_3 - 10x_2 + u_3 - x_2^3 \\ -x_3 - 2x_1 x_2 - 10u_1 x_2 \end{bmatrix};$

$\varepsilon(t) = \begin{bmatrix} \varepsilon_1(t) \\ 0 \end{bmatrix}$

With $\varepsilon_1(t) = 0.1\sin(0.1\pi t)$. Hence, one has

$$\Delta_\theta = diag\begin{bmatrix} I_1 & 1/\theta I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\theta & 0 \\ 0 & 0 & 1/\theta \end{bmatrix}$$

An observer of the form (7) has been designed to estimate x where the inputs are given by $u_1(t) = \sin(11\pi t)$, $u_2(t) = \cos(11\pi t)$ and $u_3(t) = 5\cos(2\pi t)$. Before being used by the observer, the measurements of x_1 issued from the simulation of system (29) have been corrupted by a Gaussian noisy signal with zero mean value and a variance equal to 0.02. Estimation results are reported in Figure 1 and they clearly show that the estimates provided the observers are very satisfactory in spite of the noisy measurements and the presence of the uncertain term $\varepsilon_1(t)$ which is ignored by the observer.

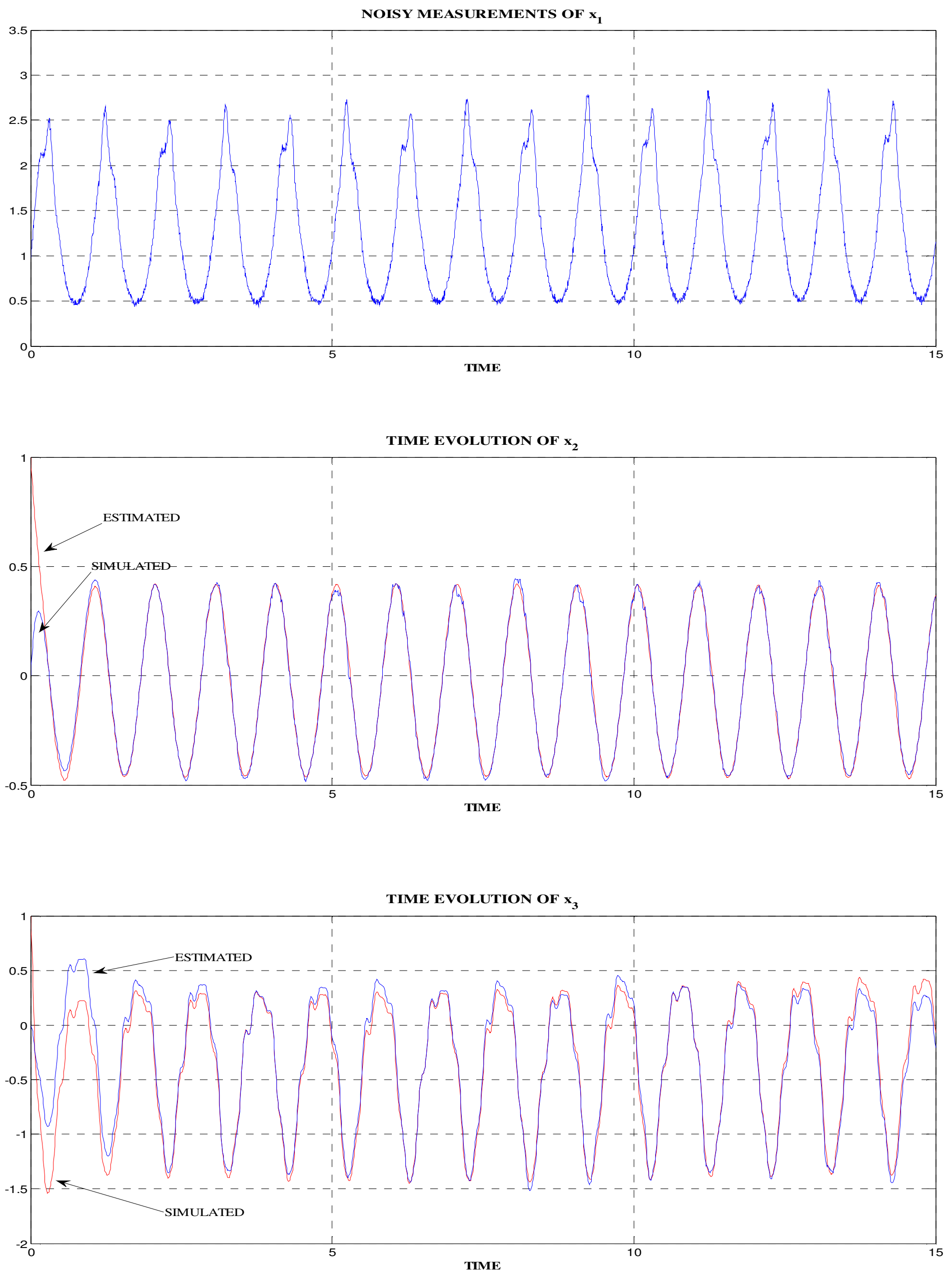

Figure 2 Estimation of the states x_2 and x_3 from noisy measurements of x_1

4. CONCLUSIONS

A high gain observer is proposed for a class of MIMO non-uniformly observable systems including uncertainties. The gain of the observer is issued from a Lyapunov differential equation and it can be tuned through the choice of a scalar design parameter. Under well defined persistent excitation conditions, it is shown that the observation error related to the proposed observer exponentially converges to zero. In the presence of uncertainties, this error is ultimately bounded and the underlying ultimate bound can be made as small as desired by choosing sufficiently high the observer design parameter. However, very high values of this parameter render more sensitive the observer to the noise measurements. As a result and as it is well known for high gain observer, the choice of the observer design parameter is a compromise between a satisfactory tracking of the state variation and a good behavior with respect to the noise measurements.

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